ISR perp. to B

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June 2006

Incoherent-scatter spectral models for modes propagating perpendicular to Earth’s magnetic field \( \mathbf{B} \) will be described.

Outline:

- **Motivation**: Why “perp. to \( \mathbf{B} \) ISR” is important?
- **ISR tutorial**: to establish a setting needed for the discussion of perp. to \( \mathbf{B} \) issues.
- 3-D modeling of collision effects — also in *Milla and Kudeki* [2006] poster.
Why “perp. to B ISR theory” is important?

Because:

- “Perp. to B” is the **natural look direction** for equatorial ISR (JRO, ALTAIR, AMISR(?)) **vertical drift measurements**.
- **ISR spectrum** is **very different** at small aspect angles $\alpha$, close to perp to $\mathbf{B}$ — familiar **double-humped** shape disappears as “overspread” scatter turns into “underspread” in $\alpha \rightarrow 0$ limit.
- Different spectral shapes correspond to different micro-physics dominant at different aspect angles.
- **ISR theory** had to be revised a number of times (over the last 40 years) in small-$\alpha$ regime to match the increasingly refined new observations coming from JRO — we are currently going through another round of revisions.
- Revisions are related to difficult issues in plasma physics concerning collisions and thus the results could have **“broader impact”**.
• ISR spectrum narrows down to “almost a delta” as $\alpha \to 0$, which is wonderful for high-precision drift measurements using “periodogram” techniques.

• A better understanding of the ISR spectrum for small-$\alpha$ opens up the possibility of density and temperature measurements that accompany drift observations — **practical impact**.

• Perp. to $\mathbf{B}$ direction is also the natural direction to observe **field-aligned plasma instabilities** — e.g., spread-F, 150-km echoes, electrojet in the equatorial ionosphere.

• Joint studies of the instabilities and surrounding ionosphere can be conducted by using perp. to $\mathbf{B}$ radar beams — another **practical reason**.

Now the tutorial …
First degree of $\alpha$:

From *Milla and Kudeki* poster [2006] showing the results of *brand new* collisional ISR spectrum calculations using a 3-D random walk code — just in case I get stuck in the tutorial and run out of time:

**Re$\{J_e(\omega)\}$ at 50 MHz:**

Electron Gordeyev integral ($N_e=1E12m^{-3}$, $T_e=1000K$, $\lambda_B=3m$)

**Re$\{J_e(\omega)\}$ at 500 MHz:**

Electron Gordeyev integral ($N_e=1E12m^{-3}$, $T_e=1000K$, $\lambda_B=0.3m$)

Unless electron Coulomb collisions are included in the theory, the narrowing of the spectrum in $\alpha \rightarrow 0$ limit is not properly modeled. — **Now the tutorial, really ...**

for 0.25 to 0 deg aspect angles the results are totally new at 50 MHz...
Thomson scatter from a single electron

Oscillating free electrons radiate like Hertzian dipoles:

\[ E_s e^{j \omega t} = -\frac{r_e}{r} E_i e^{j(\omega t - 2k r)} \]

is the electric field “backscattered” or “radiated back” from a single electron at a distance \( r \) in response to an incident field (real parts are implied in both expressions)

\[ E_i e^{j(\omega t - k r)} \]

of frequency \( \omega_o \) and wavenumber \( k_o = \frac{\omega_o}{c} = \frac{2\pi}{\lambda_o} \),

\[ r_e \equiv \frac{e^2}{4\pi \epsilon_0 m c^2} \approx 2.818 \times 10^{-15} \text{ m} \]

is a fundamental length scale known as classical electron radius.
Backscatter from a small volume of electrons

Backscattered field envelope from a small volume $\Delta V$ centered at $\mathbf{r} = r\hat{r}$ containing $P$ free electrons at an average-density of $N_o = P/\Delta V$ is the simple sum

$$E_s = - \sum_{p=1}^{N_o\Delta V} \frac{r_e}{r_p} E_{ip} e^{-j2k_0r_p} \to -\frac{r_e}{r} E_i \sum_{p=1}^{N_o\Delta V} e^{jk_\mathbf{p}}.$$  

The paraxial limit on the right is valid for $r > 4\Delta V^{2/3}/\lambda_o$ (effectively the far-field condition for an antenna of size $\Delta V^{1/3}$ and wavelength $\lambda_o/2$) while

$$k \equiv -2k_0\hat{r},$$

known as Bragg vector, is the scattered minus incident wavevector relevant to scattering volume $\Delta V$.  

Particle trajectories $r_p(t)$ and density-waves $n(k,t)$:

The scattered field varies with time as

$$E_s(t) = -\frac{r_e}{r} E_i \sum_{p=1}^{N_0\Delta V} e^{jk \cdot r_p(t)} = -\frac{r_e}{r} E_i n(k,t)$$

where

$$n(k,t) \equiv \sum_{p=1}^{N_0\Delta V} e^{jk \cdot r_p(t)}$$

is the spatial Fourier transform $\int dr \ n(r,t) \ e^{jk \cdot r}$ of

$$n(r,t) = \sum_{p=1}^{N_0\Delta V} \delta(r - r_p(t)),$$

a number density function defined for electrons with trajectories $r_p(t)$.

Note: Normalized variance

$$\frac{1}{\Delta V} \langle |n(k,t)|^2 \rangle$$

in $\Delta V \to \infty$ limit (meaning $\Delta V^{1/3} >$ a few correlation scales) is the spatial power spectrum of density fluctuations due to random trajectories $r_p(t)$.

Density space-time spectrum is (likewise) the Fourier transform of normalized auto-correlation (ACF)

$$\frac{1}{\Delta V} \langle n^*(k,t)n(k,t + \tau) \rangle$$

of $n(k,t)$ over time lag $\tau$ (see next page).
“Soft-target” power spectra

\[ E_s(t) = -\frac{r_e}{r} E_i \ n(k, t) \Rightarrow \langle |E_s(\omega)|^2 \rangle = \frac{r_e^2}{r^2} |E_i|^2 \langle |n(k, \omega)|^2 \rangle \Delta V \]

in terms of electron-density space-time spectrum

\[ \langle |n(k, \omega)|^2 \rangle \equiv \int d\tau e^{-j\omega \tau} \frac{1}{\Delta V} \langle \sum_{p=1}^{N_o \Delta V} e^{-j k \cdot r_p(t)} \sum_{p=1}^{N_o \Delta V} e^{j k \cdot r_p(t+\tau)} \rangle \]

Also the total power collected by a radar antenna with an effective aperture \( A_e \) — adding the spectrum over all frequencies \( \omega/2\pi \) and subvolumes \( \Delta V \) — is (open-bandwidth case)

\[ P_r = \int \frac{d\omega}{2\pi} \int dV \ \frac{|E_i|^2/2\eta_o}{r^2} A_e r_e^2 \langle |n(k, \omega)|^2 \rangle \quad \text{—Radar eqn.} \]

\[ \langle |n(k, \omega)|^2 \rangle \text{ is the F.T. over time lag } \tau \text{ of the normalized ACF} \]

\[ \frac{1}{\Delta V} \langle n^*(k, t) n(k, t+\tau) \rangle. \]

Above and elsewhere, angular brackets \( \langle \text{ and } \rangle \) around a random variable imply an expected value or ensemble average.
"Soft-target" power spectra

\[ E_s(t) = -\frac{r_e}{r} E_i n(k, t) \Rightarrow \langle |E_s(\omega)|^2 \rangle = \frac{r_e^2}{r^2} |E_i|^2 \langle |n(k, \omega)|^2 \rangle \Delta V \]

in terms of electron-density space-time spectrum

\[ \langle |n(k, \omega)|^2 \rangle = \int d\tau e^{-j\omega \tau} \frac{1}{\Delta V} \sum_{p=1}^{N_o\Delta V} e^{-j\mathbf{k} \cdot \mathbf{r}_p(t)} \sum_{p=1}^{N_o\Delta V} e^{j\mathbf{k} \cdot \mathbf{r}_p(t+\tau)} \]

\[ = N_o \int d\tau e^{-j\omega \tau} \langle e^{j\mathbf{k} \cdot \Delta \mathbf{r}} \rangle \equiv \langle |n_{te}(k, \omega)|^2 \rangle, \]

assuming that electrons follow random trajectories with independent displacements \( \Delta \mathbf{r} \equiv \mathbf{r}(t + \tau) - \mathbf{r}(t) \). But the assumption is not valid, and its direct consequence (in thermal equilibrium)

\[ \langle |E_s(\omega)|^2 \rangle \propto \langle |n_{te}(k, \omega)|^2 \rangle \propto \int d\tau e^{-j\omega \tau} \langle e^{j\mathbf{k} \cdot \mathbf{v} \tau} \rangle \propto e^{-\frac{\omega^2}{2k^2C_e^2}}, \]

a Gaussian radar spectrum of a width \( \propto \) electron thermal speed \( C_e \) — original expectation of Gordon [1958] — is not observed.

because of "collective effects" due to polarization fields (spectra for 41 MHz observations of Bowles, 1958).
Including the “collective effects”

If there were no collective effects

- spectrum of electron and ion density fluctuations in the plasma would be

\[
\langle |n_{te,i}(k, \omega)|^2 \rangle \equiv N_o \int_{-\infty}^{\infty} d\tau e^{-j\omega \tau} \langle e^{j k \cdot \Delta r_{e,i}} \rangle,
\]

with
- \(N_o\) average plasma density
- \(\Delta r_{e,i} \equiv r_{e,i}(t + \tau) - r_{e,i}(t)\) independent particle displacements

- of course there would also be random current densities \(\frac{\omega}{k} e(n_{ti} - n_{te})\) and space-charge fluctuations \(e(n_{ti} - n_{te})\) satisfying the plane-wave continuity equation across \(k-\omega\) space

Collective effects come into play because space-charge \(\propto n_{ti} - n_{te}\) requires (via Poisson’s equation) a longitudinal electric field \(E\) (parallel to \(k\)) which, in turn, drives additional currents \(\sigma_e E\) and \(\sigma_i E\) to force the total current, including the
displacement current \( j \omega \epsilon_o E \), to vanish; thus

\[
(j \omega \epsilon_o + \sigma_e + \sigma_i)E + \frac{\omega}{k} e(n_{ti} - n_{te}) = 0
\]

so that Ampere’s law applied to longitudinal (space-charge) waves (influenced by collective effects) is satisfied.

One of the solutions of this “KCL equation” — obtained with the aid of an equivalent circuit model shown below — is the electron-density wave amplitude

\[
n(k, \omega) = \frac{(j \omega \epsilon_o + \sigma_i)n_{te}(k, \omega)}{j \omega \epsilon_o + \sigma_e + \sigma_i} + \frac{\sigma_e n_{ti}(k, \omega)}{j \omega \epsilon_o + \sigma_e + \sigma_i},
\]

a weighted sum that can be interpreted in terms of “shielded” versions of thermally driven densities \( n_{te} \) and \( n_{ti} \) — shielding reduces the overall space-charge by a factor \(|1 + \chi_e + \chi_i| \gg 1\), where \( \chi_{e,i} \equiv \sigma_{e,i}/j \omega \epsilon_o \) are electron- and ion-susceptibilities.
We have expressed the actual electron density fluctuation in the plasma in terms of independent random variables $n_{te}$ and $n_{ti}$; thus, upon squaring and averaging the expression we find that electron density spectrum

$$\langle |n(k, \omega)|^2 \rangle = \frac{|j \omega \epsilon_o + \sigma_i|^2 \langle |n_{te}(k, \omega)|^2 \rangle}{|j \omega \epsilon_o + \sigma_e + \sigma_i|^2} + \frac{|\sigma_e|^2 \langle |n_{ti}(k, \omega)|^2 \rangle}{|j \omega \epsilon_o + \sigma_e + \sigma_i|^2},$$

a sum of electron- and ion-lines, proportional to $\langle |n_{te,i}(k, \omega)|^2 \rangle$, respectively.

The spectrum formula above is a very general result which is valid with any type of velocity distribution (i.e., Maxwellian or not). It can be modified in a straightforward way to treat the multi-ion case. It is also valid in magnetized plasmas in electrostatic approximation — i.e., for nearly longitudinal modes with phase speeds $\omega/k \ll c$ — with $\sigma_{e,i} = \sigma_{e,i}(k, \omega)$ denoting the longitudinal component of particle conductivities. However, to use it we need accurate knowledge of all $\sigma_{e,i}(k, \omega)$.

Fortunately, there are some wonderful links between conductivities $\sigma_{e,i}(k, \omega)$ and $e^{jk \cdot \Delta r}$-statistics of particles that we can use.
First, according to generalized Nyquist noise theorem [e.g., Callen and Greene, 1952], mean-squared particle current due to random thermal motions is (in the particle frame)

\[
\frac{\omega^2}{k^2} e^2 \langle |n_{te,i}(k, \omega)|^2 \rangle = 2K T_{e,i} \text{Re}\{\sigma_{e,i}(k, \omega)\}
\]

per unit bandwidth, per species, so long as each species is in thermal equilibrium (i.e., have a Maxwellian velocity distribution) at a temperature\(^1\) \(T_{e,i}\).

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\(^1\)When \(T_e = T_i = T\), i.e., in case of full thermal equilibrium, \(\frac{\omega^2}{k^2} e^2 \langle |n(k, \omega)|^2 \rangle = 2K T \text{Re}\{\sigma_t(k, \omega)\}\) with \(\sigma_t\) representing the Thevenin admittance of the equivalent circuit looking into the opened electron branch.
Second, as a consequence of \textit{causality}, imaginary part $\text{Im}\{\sigma_{e,i}(\mathbf{k}, \omega)\}$ of conductivity $\sigma_{e,i}(\mathbf{k}, \omega)$ is the \textit{Hilbert transform} of $\text{Re}\{\sigma_{e,i}(\mathbf{k}, \omega)\}$, a general rule known as \textit{Kramers-Kronig relation} which applies to all Fourier transforms of causal signals that vanish for $t < 0$.

\textit{The upshot is}, in a plasma in thermal equilibrium, all parameters needed to compute the electron density spectrum can be deduced from

$$\langle e^{j\mathbf{k} \cdot \Delta r_{e,i}} \rangle,$$

\textit{characteristic functions} of particle displacements $\Delta r_{e,i}$ in the absence of collective effects. We will call them “single particle ACF’s” in the following discussions.
What we have seen so far was *distilled* from a number of different approaches to incoherent scatter problem worked out during the 1960’s:

- *Farley* and co-writers derive $\sigma_{e,i}(k, \omega)$ from plasma kinetic theory (Vlasov equation) and then use the Nyquist formula to obtain $\langle |n_{te,i}(k, \omega)|^2 \rangle$.
- *Fejer* does both calculations independently, not using (but effectively re-deriving) the Nyquist formula.
- *Woodman* takes yet another approach, including steps involved in the proof of the generalization of Nyquist theorem by *Callen and Greene* [1952], but not using Nyquist’s formula explicitly.
- *Hagfors* and collaborators first calculate $\langle |n_{te,i}(k, \omega)|^2 \rangle$ from $\langle e^{jk\cdot\Delta r_{e,i}} \rangle$ and then “dress” the particles making up $n_{te,i}(k, \omega)$ with $\sigma_{e,i}(k, \omega)$ dependent “shields” to obtain the expression for electron density spectrum — Nyquist formula is effectively re-derived.

These pioneers have handed us (the current generation of ISR users) a ...
"Standard Model"

\[ J_s(\omega) \equiv \int_0^\infty d\tau \, e^{-j\omega \tau} \langle e^{jk \cdot \Delta r_s} \rangle \quad \text{Gordeyev integral, a 1-sided F.T.} \]

for species \( s \) (\( e \) or \( i \) for the single-ion case), and use

\[
\frac{\langle |n_{ts}(k, \omega)|^2 \rangle}{N_o} = 2\text{Re}\{J_s(\omega_s)\} \quad \text{and} \quad \frac{\sigma_s(k, \omega)}{j\omega \epsilon_o} = \frac{1 - j\omega_s J_s(\omega_s)}{k^2 h_s^2},
\]

where \( \omega_s \equiv \omega - k \cdot V_s \) is Doppler-shifted frequency in the radar frame due to mean velocity \( V_s \) of the species and \( h_s = \sqrt{\epsilon_o K T_s / N_o e^2} \) is the corresponding Debye length. In terms of above definitions, electron density spectrum of a stable Maxwellian plasma is

\[
\langle |n(k, \omega)|^2 \rangle = \frac{|j\omega \epsilon_o + \sigma_i|^2 \langle |n_{te}(k, \omega)|^2 \rangle}{|j\omega \epsilon_o + \sigma_e + \sigma_i|^2} + \frac{\sigma_e^2 \langle |n_{ti}(k, \omega)|^2 \rangle}{|j\omega \epsilon_o + \sigma_e + \sigma_i|^2}.
\]

The model takes care of macrophysics of incoherent scatter — microphysics details need to be addressed within single particle ACF’s \( \langle e^{jk \cdot \Delta r_s} \rangle \).
Single particle ACFs \( \langle e^{jk \cdot \Delta r} \rangle \equiv \langle e^{jk \cdot (r(t+\tau) - r(t))} \rangle \)

are the centerpiece of Standard Model — their Fourier transforms or Gordeyev integrals (obtained numerically in most cases) provide us with the conductivities and spectra of all species in a plasma (in thermal equilibrium).

In general, if \( \Delta r \), component of \( \Delta r \) along \( k \), is a Gaussian random variable, then

\[
\langle e^{jk \cdot \Delta r} \rangle = e^{-\frac{1}{2} k^2 \langle \Delta r^2 \rangle}.
\]

**Example:** In a *non-magnetized plasma* particles move along *straight line trajectories* (in between collisions) with velocities \( v \) and thus

\[
\Delta r = v \tau;
\]

hence

\[
\langle \Delta r^2 \rangle = \langle v^2 \rangle \tau^2 = C^2 \tau^2,
\]

for a Maxwellian (required by Standard Model ) distributed \( v \) along \( k \) with an rms speed \( \langle v^2 \rangle^{1/2} = \sqrt{KT/m} \equiv C \). Thus, in a non-magnetized plasma

\[
\langle e^{jk \cdot \Delta r} \rangle = e^{-\frac{1}{2} k^2 C^2 \tau^2}
\]

so long as “collision frequency” \( \nu \) is small compared to \( kC \) — i.e., if an average particle moves a distance of many wavelengths \( \frac{2\pi}{k} \) in between collisions.
In a collisional plasma $\langle \Delta r^2 \rangle = C^2 \tau^2$, special for free-streaming particles, stays valid until “first collisions” take place at $\tau \sim \nu^{-1}$. For $\nu \tau \gg 1$, collisional random walk process leads to $\langle \Delta r^2 \rangle \propto \tau$ instead of $\tau^2$, and more specifically, over all $\tau$,

$$\langle \Delta r^2 \rangle = \frac{2C^2}{\nu^2} (\nu \tau - 1 + e^{-\nu \tau}) \Rightarrow ACF = \begin{cases} e^{-\frac{1}{2}k^2 C^2 \tau^2}, & \nu \ll kC \\ e^{-k^2 C^2 \nu \tau}, & \nu \gg kC \end{cases}$$

if a Brownian-motion model is adopted for collisions. Using the high-collision approximation above (which is not sensitive to the choice collision model, e.g., Brownian, BGK, etc.) a Lorentzian shaped electron density spectrum pertinent to D-region altitudes can be easily obtained (mainly the “ion-line”):

$$\frac{\langle |n(k, \omega)|^2 \rangle}{N_o} \approx \frac{2k^2 D_i}{\omega^2 + (2k^2 D_i)^2}$$

in $kh \ll 1$ limit (wavelength larger than Debye length) with $D_i \equiv C_i^2 / \nu_i = KT_i / m_i \nu_i$, ion diffusion coefficient.

Collisional D-region spectra from JRO:

Chau and Kudeki [2006]
However, a complete D-region model should require a multi-ion formulation including negative ions [e.g., Mathews, 1978].

**Generalizations:**

Using the above result, it is easy to show that

\[ \langle |n(k)|^2 \rangle \equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \langle |n(k, \omega)|^2 \rangle = \frac{N_o}{2}, \]

which is in fact true in general — i.e., for all types of plasmas with or without collisions and/or DC magnetic field — so long as \( T_e = T_i \) and \( kh \ll 1 \).

This result in turn leads to a well-known volumetric radar cross-section formula for incoherent backscatter (valid under the same conditions):

\[ 4\pi r_e^2 \langle |n(k)|^2 \rangle = 2\pi r_e^2 N_o. \]

Only for \( kh \gg 1 \) we obtain \( \langle |n(k)|^2 \rangle = N_o. \)

Notice aspect angle dependent errors from 200 to 300 km where \( T_e > T_i. \)
Plasma with a DC magnetic $B_o$

$$\langle e^{jk\cdot \Delta r} \rangle = \langle e^{j(k\parallel \Delta r + k\perp \Delta p)} \rangle = \langle e^{jk\parallel \Delta r} \times e^{jk\perp \Delta p} \rangle,$$

where $\Delta r$ and $\Delta p$ are particle displacements along and perp to $B_o$ on $k-B_o$ plane.

Assuming independent Gaussian random variables $\Delta r$ and $\Delta p$, we can write

$$\langle e^{jk\cdot \Delta r} \rangle = e^{-\frac{1}{2}k^2\langle \Delta r^2 \rangle} \times e^{-\frac{1}{2}k^2\langle \Delta p^2 \rangle}$$

in analogy with non-magnetized case. The assumptions are valid in the absence of collisions, in which case

$$\langle \Delta r^2 \rangle = C^2 \tau^2 \quad \text{and} \quad \langle \Delta p^2 \rangle = \frac{4C^2}{\Omega^2} \sin^2 (\Omega \tau / 2),$$

where $\Omega$ is the gyro-frequency and periodic $\langle \Delta p^2 \rangle$ is fairly easy to confirm in terms of circular orbits with periods $2\pi/\Omega$ and mean radii $\sqrt{2C}/\Omega$. 

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Thus,

\[ \langle e^{j\mathbf{k} \cdot \Delta \mathbf{r}} \rangle = e^{-\frac{1}{2}k^2C^2\tau^2} \times e^{-\frac{2k^2}{\Omega^2}\sin^2(\Omega \tau/2)} \]

**Spectrum examples:**

Large \( \alpha \): the usual ion-line

Small \( \alpha \): "electron-line" with a reduced width

Note, the ACF above becomes periodic and the associated spectra are singular (with delta functions) in \( k_{||} \rightarrow 0 \) limit. Singularities are not observed in practice and it was recognized early on to include *Coulomb collisions* — electrostatic interactions of nearby particles within a Debye length not covered by collective effects — in the theory [Farley, 1964]. Examples above were obtained with collisional equations of Woodman [1967] that includes ion-ion collisions.
A collisional/magnetized model, consistent with independent and Gaussian $\Delta r$ and $\Delta p$ assumptions, is obtained with

$$
\langle \Delta r^2 \rangle = \frac{2C^2}{\nu^2} (\nu \tau - 1 + e^{-\nu \tau}),
$$

$$
\langle \Delta p^2 \rangle = \frac{2C^2}{\nu^2 + \Omega^2} (\cos(2\gamma) + \nu \tau - e^{-\nu \tau} \cos(\Omega \tau - 2\gamma)),
$$

where $\gamma \equiv \tan^{-1} \nu / \Omega$ — first derived by Woodman [1967] using what is effectively a Brownian motion model in the presence of $\mathbf{B}_0$. In perp to $\mathbf{B}_0$ limit:

$$
\langle e^{jk \cdot \Delta r} \rangle \rightarrow e^{-\frac{k^2 C^2}{\Omega^2 + \nu^2} (\cos(2\gamma) + \nu \tau - e^{-\nu \tau} \cos(\Omega \tau - 2\gamma))}
$$

is non-periodic, ion resonances go-away, electron-line is broadened:

**Ion-line gyro-resonances are suppressed.**

Effective (velocity averaged) Coulomb collision frequencies for a singly ionized plasma (after Spitzer, 1958):

$$
\nu_e = \frac{4\sqrt{2\pi} N_i e^4 \ln(12\pi N_e h_e^3)}{3(4\pi\epsilon_0)^2 \sqrt{m_e T_e^3}} \propto \frac{N_i}{T_e^{3/2}},
$$

$$
\nu(v) = \frac{4\pi N_i e^4 \ln \Lambda}{(4\pi\epsilon_0)^2 m_e v^3}, \nu_i = \sqrt{\frac{m_e T_e^3}{2m_i T_i^3} \nu_e}
$$

... but what we wanted at that point was spectral narrowing, not broadening !!!
While electron collisions cause spectral broadening at $\alpha = 0$ (by enabling cross-field diffusion), their effect turns out to be in the opposite direction at small but non-zero $\alpha$ because of parallel-dynamics:

$$
\frac{k^2 C^2}{\nu^2} e^{-\frac{1}{2}k^2 C^2 \tau^2} \left(\nu \tau - 1 + e^{-\nu \tau}\right) \rightarrow \left\{
\begin{array}{ll}
\sim e^{-\frac{1}{2}k^2 C^2 \tau^2}, & \nu \ll kC \quad \text{— free streaming} \\
\sim e^{-\frac{k^2 C^2}{\nu \tau}}, & \nu \gg kC \quad \text{— diffusion limit}
\end{array}
\right.
$$

- first line above, valid at larger $\alpha$ or $k$, accounts for the usual narrowing of electron-line with decreasing $\alpha$ in the absence of collisions,
- the second line, valid for smaller $\alpha$, predicts additional narrowing due to collisions, just like in D-region narrowing of ion-line with increasing $\nu$ — basically, collisions impede motion along $B_0$, lengthening correlation times and narrowing the corresponding spectra. However, the narrowing effect is still quite weak at $\alpha \approx 2^\circ$ when the Brownian collision model is used.

Sulzer and Gonzalez [1999] conjectured that a proper treatment of electron Coulomb collisions should do the job — i.e., eliminate non-physical results of $T_e < T_i$ inferred from JRO F-region data taken at $\alpha \approx 2^\circ$ [e.g., Pingree, 1990] — and proved their point by simulating the Coulomb collision process for electrons.

Figure 3. Island area $T_e/T_i$, for selected heights on June 16-17, 1998, using the 39 antenna position (adapted from Pin-
gree [1993]).

from Aponte et al. [2001], illustrating $T_e/T_i < 1$. 

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The Brownian motion model is based on an assumption of constant collision/diffusion coefficients in a governing “Langevin equation” — a 1st order stochastic differential equation governing electron velocity \( v(t) \) [e.g., Gillespie, 1996] — whereas, “in reality”, the coefficients for Coulomb collisions are \( v(t) \) dependent. Thus in reality the equation for \( v(t) \) is non-linear, causing the statistics of \( v(t) \) and its time integral \( \Delta r \) to become non-Gaussian. To address this difficulty and explore its implications, Sulzer and Gonzalez [1999] computed the electron ACF \( \langle e^{i\mathbf{k} \cdot \Delta r} \rangle \) numerically using a Monte Carlo approach:

The positions and velocities \( \mathbf{r}(t) \) and \( \mathbf{v}(t) \) of simulated electron motions were updated at \( \Delta t \) intervals with increments

\[
\Delta \mathbf{r} = \mathbf{v} \Delta t
\]

and

\[
\Delta \mathbf{v} = \mathbf{K} \Delta t + \delta \mathbf{v},
\]

where \( \delta \mathbf{v} \) is a Gaussian random variable with \( \mathbf{v} \) and \( \Delta t \) dependent moments — derived specifically for Coulomb collisions by Rosenbluth et al. [1957] and others dating back to Chandrasekhar [1942] — and \( \mathbf{K} \) is a deterministic external force per unit mass.
The update equations above constitute jointly the Langevin equation of a multivariate Markov process (non-linear and non-Gaussian) consisting of the components of \( \mathbf{r}(t) \) and \( \mathbf{v}(t) \). Estimates of ACF \( \langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}} \rangle \) were formed as the inverse Fourier transform of power spectra of synthesized time-series \( e^{j\mathbf{k}\cdot\mathbf{r}(t)} \). In spectrum calculations standard FFT methods were employed, just like in radar data analysis. A library of Gordyev integrals derived from simulated \( \langle e^{j\mathbf{k}\cdot\Delta\mathbf{r}} \rangle \) is used ultimately for density spectrum calculations.

\[
\begin{align*}
\frac{d(\Delta v)}{dt} &= -A_D f_j \left( 1 + \frac{m_e}{m_f} \right) G(l_f v) \\
\frac{d((\Delta v)^2)}{dt} &= \frac{A_D}{v} G(l_f v) \\
\frac{d((\Delta v)^2)}{dt} &= \frac{A_D}{v} \{ G(l_f v) - G(l_f v) \}
\end{align*}
\]

where

\[
A_D = \frac{a_f e^4 \ln \Lambda}{2\pi m_f^2 v^2}.
\]

\( A_D \) differs from the definition of Spitzer [1962] only in that the units have been changed from cgs to MKS. Also \( f_j = m_f/2kT \); \( f \) designates the field particles, the ones being collided with, either \( e \) for electron or \( i \) for ion, while \( m_e \) refers to the test particle which is always an electron. We have assumed that all species have a charge number of 1. Finally,

\[
G(x) = \frac{\phi(x) - x\phi'(x)}{2x^2}
\]

where

\[
\phi(x) = \frac{2}{\pi} \int_0^e e^{-y^2} dy
\]
Simulated versus collisionless spectra show considerable differences at small aspect angles \( \alpha \), enough to correct the \( T_e/T_i \) problem at \( \alpha \approx 2^\circ \) measurements.

Woodman [2004] re-examined the Brownian model and — agreeing with the main findings of Sulzer and Gonzalez — developed an empirical collision-frequency model \( \nu_e = \nu_e(\alpha) \) that gives the best fit of Brownian spectra to Sulzer and Gonzalez [1999] simulation results.
Woodman model $\nu_e = \nu_e(\alpha)$ effectively extrapolates the Sulzer and Gonzalez simulation results from $\alpha = 0.25^\circ$ to $0^\circ$ and is convenient to use in place of the Sulzer and Gonzalez Gordeyev library.

Fig. 6. Same as in Fig. 5 but both the collision frequency and the angle (actually $\sin \theta$) are normalized with respect to $\nu_0$ and $\sin \theta_c$, respectively. The dotted line is a cubic regression fit representing Eq. (14). The points corresponding to $6^\circ$ (right most in any sequence) are not included in the fit.

with $\sin \theta_c = \frac{\lambda}{\ell} = \frac{\lambda \nu_e}{C_e}$.
Highlights of Milla and Kudeki [2006] poster:

- Studies described in *Milla and Kudeki* [2006] aim to:
  - explore the goodness of Woodman’s $\nu_\text{e} = \nu_\text{e}(\alpha)$ model in $\alpha \to 0$ limit and at radar wavelengths other than 3 m for which the model was developed,
  - improve the model if needed

- by using the same methodology as *Sulzer and Gonzalez* [1999], except for:
  - include finite gyro-radius effects by doing 3-D computations of particle orbits instead of 1-D (parallel $B_0$) computations
  - extend the computations all the way to $\alpha = 0$.

- Initial results:
  - agree with *Sulzer and Gonzalez* [1999] results except for a minor offset (~10% near spectral peak as $\alpha \to 0.25^\circ$) the source of which was identified in Sulzer’s code — a typo that replaces some $\sqrt{2}$ by $\sqrt{\pi/2}$.
  - Woodman’s $\nu_\text{e} = \nu_\text{e}(\alpha)$ model inherits the offset just described but otherwise agrees with the simulated spectrum variations as $\alpha \to 0$.
  - Woodman’s $\nu_\text{e} = \nu_\text{e}(\alpha)$ model needs “retuning” at other radar wavelengths (that is, other than at 50 MHz)
Last 1000 millidegrees of $\alpha$:

$\text{Re}\{J_e(\omega)\}$ at 50 MHz:

Electron Gordeyev integral ($N_e=1 \times 10^3 m^{-3}$, $T_e=1000K$, $\lambda_B=3m$)

Note how $\langle |n_{te}(k, \omega)|^2 \rangle \propto \text{Re}\{J_e(\omega)\}$ narrows down more rapidly at 50 MHz than at 500 MHz as approaches $\alpha = 0$.

A “testable” prediction using a pair of ISR’s — VHF and UHF — near the magnetic equator: ALTAIR or, alternatively, JRO/AMISR combo.
Last 500 millidegrees of $\alpha$:

$\text{Re}\{J_e(\omega)\}$ at 50 MHz:

$\text{Re}\{J_e(\omega)\}$ at 500 MHz:

Note how $\langle |n_{te}(k, \omega)|^2 \rangle \propto \text{Re}\{J_e(\omega)\}$ narrows down more rapidly at 50 MHz than at 500 MHz as approaches $\alpha = 0$.

A “testable” prediction using a pair of ISR’s — VHF and UHF — near the magnetic equator: ALTAIR or, alternatively, JRO/AMISR combo.

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More detailed comparisons show that:

1. Brownian motion based ion-Gordeyev integrals match ion Monte Carlo results very well using for $\nu_i$ the effective Coulomb collision frequency (due to Spitzer) for ions given earlier. This finding is consistent with the success of Woodman [1967] theory in showing the absence of ion gyro-resonance effects.

2. Monte Carlo results for electron displacements $\Delta p$ transverse to $B_o$ exhibit a Gaussian $\Delta p$ with a variance $\langle \Delta p^2 \rangle$ matching well the Brownian motion model using $\frac{5}{3} \nu_e$ for $\nu$, where $\nu_e$ is the Spitzer collision frequency for electrons. The factor $5/3$ is independent of $N_e$ and $T_e$, and is likely to be due to electron-electron collisions not included in Spitzer's $\nu_e$.

3. Monte Carlo results show that $\Delta r$ for electrons is a non-Gaussian random variable for $\tau \sim \nu_e^{-1}$ and Gordeyev integrals obtained from Brownian motion versus Monte Carlo calculations do not match except in $\alpha \to 0$ limit.

4. Modified Brownian model of Woodman [2004] shows a reasonable agreement with the simulations at 50 MHz (3 m radar) except for a minor offset, but it requires adjustments at higher probing frequencies such as 500 MHz (30 cm radar).

5. Electron-collision effects are less pronounced for a 30 cm radar than for 3 m radar (as expected), but still the effects cannot be neglected at small $\alpha$. 
At very very small aspect angles, 1 and 5 milli-degrees ($T_{e,i} = 1000 \, \text{K}, \, O^+$):
At perp. to $B$ (collisionless is a $\delta$ here) and at $0.5^\circ$ off-perp.:

**ISR Spectrum $- \lambda_B = 3 \, m$, $\alpha = 0^\circ$**

**ISR Spectrum $- \lambda_B = 3 \, m$, $\alpha = 0.5^\circ$**

**ISR Spectrum $- \lambda_B = 0.3 \, m$, $\alpha = 0^\circ$**

**ISR Spectrum $- \lambda_B = 0.3 \, m$, $\alpha = 0.5^\circ$**
Conclusions

- *Sulzer and Gonzalez* [1999] simulations of electron-Coulomb collisions and their impacts on ISR spectra at small aspect angles were — *for all practical purposes* — confirmed by our simulations, which were conducted with independent software and algorithms using a 3-D setting (instead of 1-D).

- The extension of the simulations to $\alpha = 0$ has shown that *Woodman* [2004] semi-empirical model works well at 50 MHz except for the need for a minor correction of a minor error inherited from *Sulzer and Gonzalez* [1999] simulations.

- The extension of simulations from 50 MHz to 500 MHz have provided the information to generalize the semi-empirical model for use over a range of practical ISR frequencies.

- We have now a working small-$\alpha$ spectral theory to subject it to further *experimental tests* and attempt inversions of measured ISR spectra at small-$\alpha$ for densities and temperatures based on the new model.

- We are optimistic that $T_e$ and $T_i$ can be estimated by using both *spectral* and *cross-spectral* data — north-south baseline cross-spectra are sensitive to $T_e/T_i$ dependent “aspect sensitivity” of incoherent scattered signal.
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