Nonstationary Time Series

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- What is a non stationary time series?

- Multitaper spectral method

- Ondelettes (Wavelets) analysis

- Discussions / Conclusions

Multi-taper Spectral Method
(Thompson, 1982)

<u>Purpose:</u> model a finite time serie

$$\{x(i)\}_{i=0,N-1}$$

by finite sum of sinusoids of angular frequencies

$$\{\omega(j)\}_{j=0,P-1}$$

<u>Optimal choice of tapers:</u>

assume $x(t) = \mu\, e^{i\hat{\omega}t}$ $\quad t = 0, N-1$

let $w(t)$ be a taper of length N

then
$$y(\omega) = \sum_{t=0}^{N-1} x(t)\, w(t)\, e^{-i\omega t} \qquad (1)$$

is the Discrete Fourier Transform of the tapered signal

Taper should be choosen to maximize

$$F(w) = \frac{\int_{\hat{\omega}-\Omega}^{\hat{\omega}+\Omega} |y(\omega)|^2\, d\omega}{\int_{-\pi}^{\pi} |y(\omega)|^2\, d\omega} \qquad (2)$$

(1) and (2) $\Rightarrow$ $\quad F(w) = \dfrac{\vec{w}\cdot A\cdot\vec{w}}{\vec{w}\cdot\vec{w}}$

with A being $N \times N$ symmetric positive definite Matrix

$$A_{ij} = \begin{cases} \dfrac{\sin \Omega (i-j)}{\pi (i-j)} & \text{if } i \neq j \\ \dfrac{\Omega}{\pi} & \text{if } i = j \end{cases}$$

Maximize F with respect to $\vec{w}$ is equivalent to solve the eigenvalue problem:

$$A \cdot \vec{w} = \lambda \vec{w}$$

The Eigenvectors are called Discrete Prolate Spheroidal Sequences (DPSS)

Eigenvectors ordered by their values

$$1 > \lambda_0 > \cdots \lambda_{N-1} > 0$$

$1 - \lambda_k$ = give the fraction of the total energy of the k^{th} window outside the interval $(\hat{\omega} - \Omega, \hat{\omega} + \Omega)$

Generally around 4-5 DPSS used in analysis: those for which $\lambda \sim 1$

<u>Harmonic analysis:</u>

assume $x(t) = \mu e^{i \hat{\omega} t} + e(t)$

where $e(t)$: other sinusoids + white noise let k be the DPSS number ($k=0,\dots,K-1$). Define:

$$y_k(\omega) = \sum_{t=0}^{N-1} x(t)\, w_k(t)\, e^{i\omega t}$$

$$U_k(\omega) = \sum_{t=0}^{N-1} w_k(t)\, e^{i\omega t}$$

$$e_k(\omega) = y_k(\omega) - \mu \sum_{t=0}^{N-1} w_k(t)\, e^{i(\hat{\omega}-\omega)t}$$

so one has $e_k(\hat{\omega}) = y_k(\hat{\omega}) - \mu\, U_k(0)$

$\Rightarrow$ We want to minimize:

$$\nu = \sum_{k=0}^{K-1} |\, e_k(\hat{\omega})|^2 \quad \text{as a function of } \mu$$

obtained writing $\dfrac{\partial \nu}{\partial \mu} = 0$ at $\omega = \hat{\omega}$

give an estimate $\hat{\mu}$ of μ:

$$\boxed{\hat{\mu}(\hat{\omega}) = \frac{\sum_{k=0}^{K-1} U_k^*(0)\, y_k(\hat{\omega})}{\sum_{k=0}^{K-1} |U_k(0)|^2}}$$

Statistical significance:

One can compare, at a given $\hat{\omega}$, explained and unexplained variances by the estimate $\hat{\mu}$.
Then Fisher test can be applied:

explained : $\theta = |\hat{\mu}(\hat{\omega})|^2 \sum_{k=0}^{K-1} |U_k(0)|^2$

Unexplained: $\psi = \sum_{k=0}^{K-1} |y_k(\hat{\omega}) - \hat{\mu}(\hat{\omega}) U_k(0)|^2$

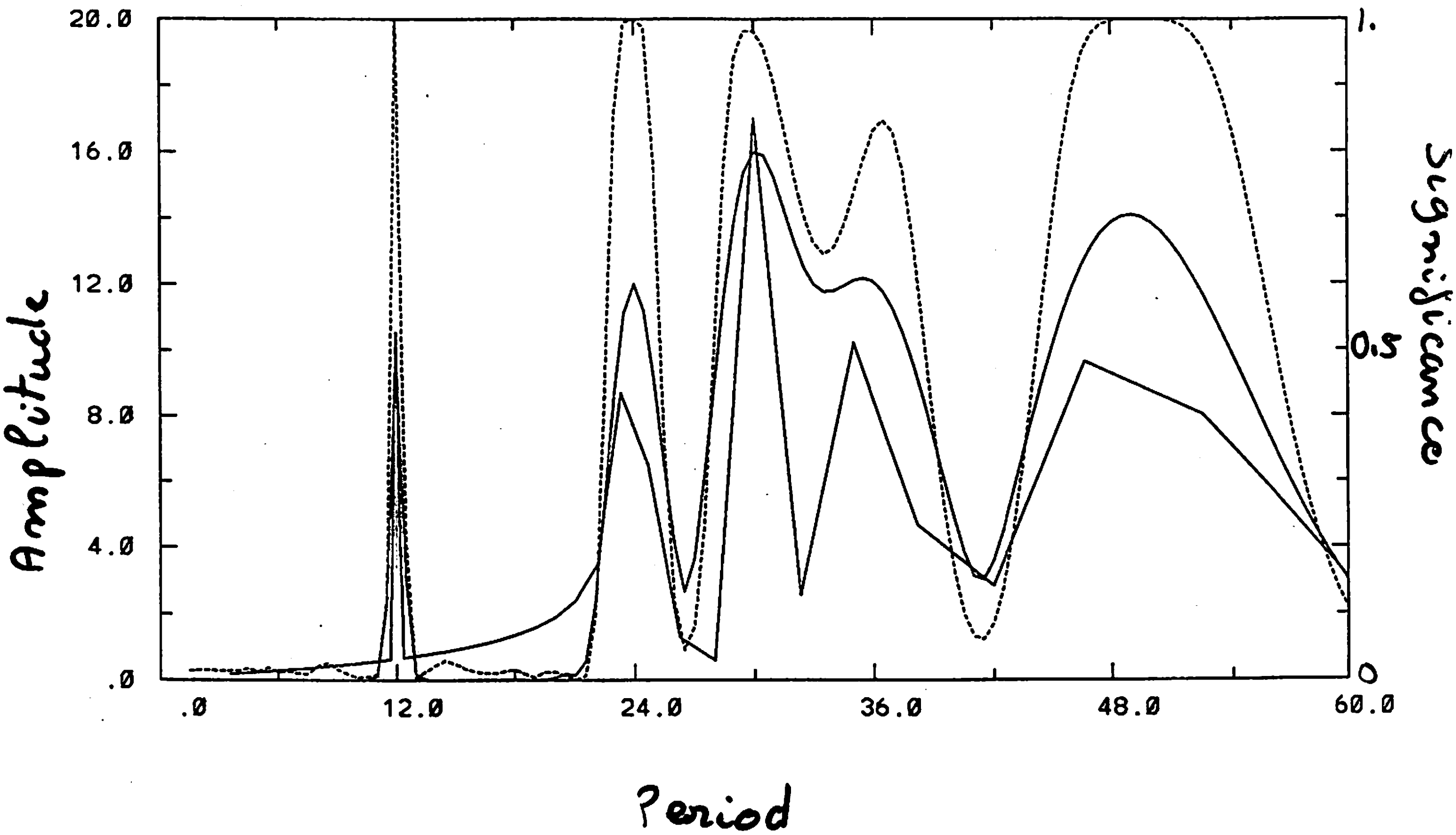
then $F(\hat{\omega}) = (K-1)\dfrac{\theta}{\psi}$ follows Fisher law with 2, and $2(K-1)$ degrees of freedom

Serie $y = 10\cos\left(\frac{2\pi t}{12} + \frac{\pi}{3}\right) + 12\sin\left(\frac{2\pi t}{24}\right) + 16\sin\left(\frac{2\pi}{30}t + \frac{\pi}{4}\right) + 12\sin\left(\frac{2\pi}{36}t + \frac{\pi}{2}\right) + 14\sin\left(\frac{2\pi}{49}t + \frac{3\pi}{5}\right)$

$t = 30$ to 450

NBS DE POINTS= 420 VALEUR PROPRE MIN= .977

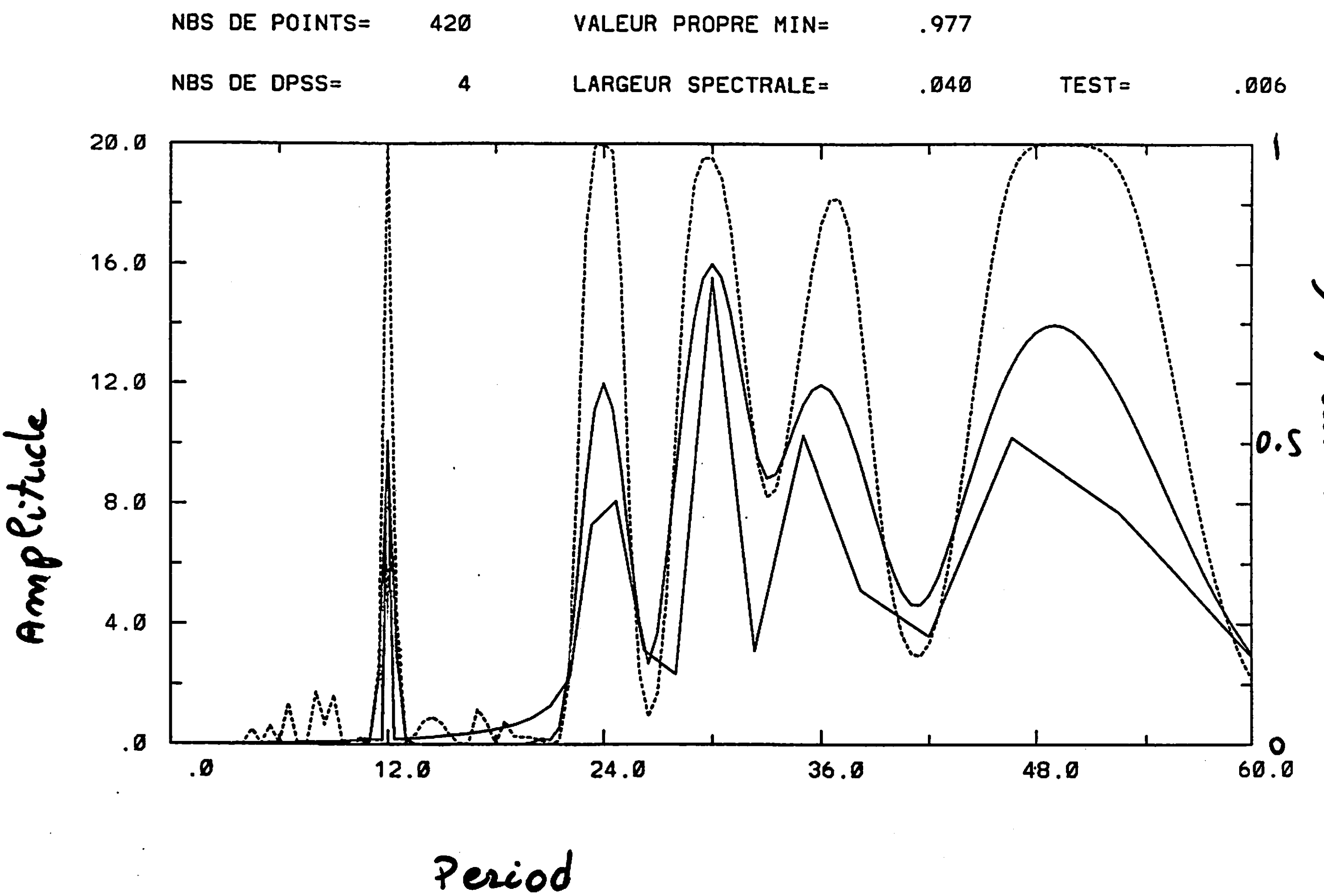
NBS DE DPSS= 4 LARGEUR SPECTRALE= .040 TEST= .006



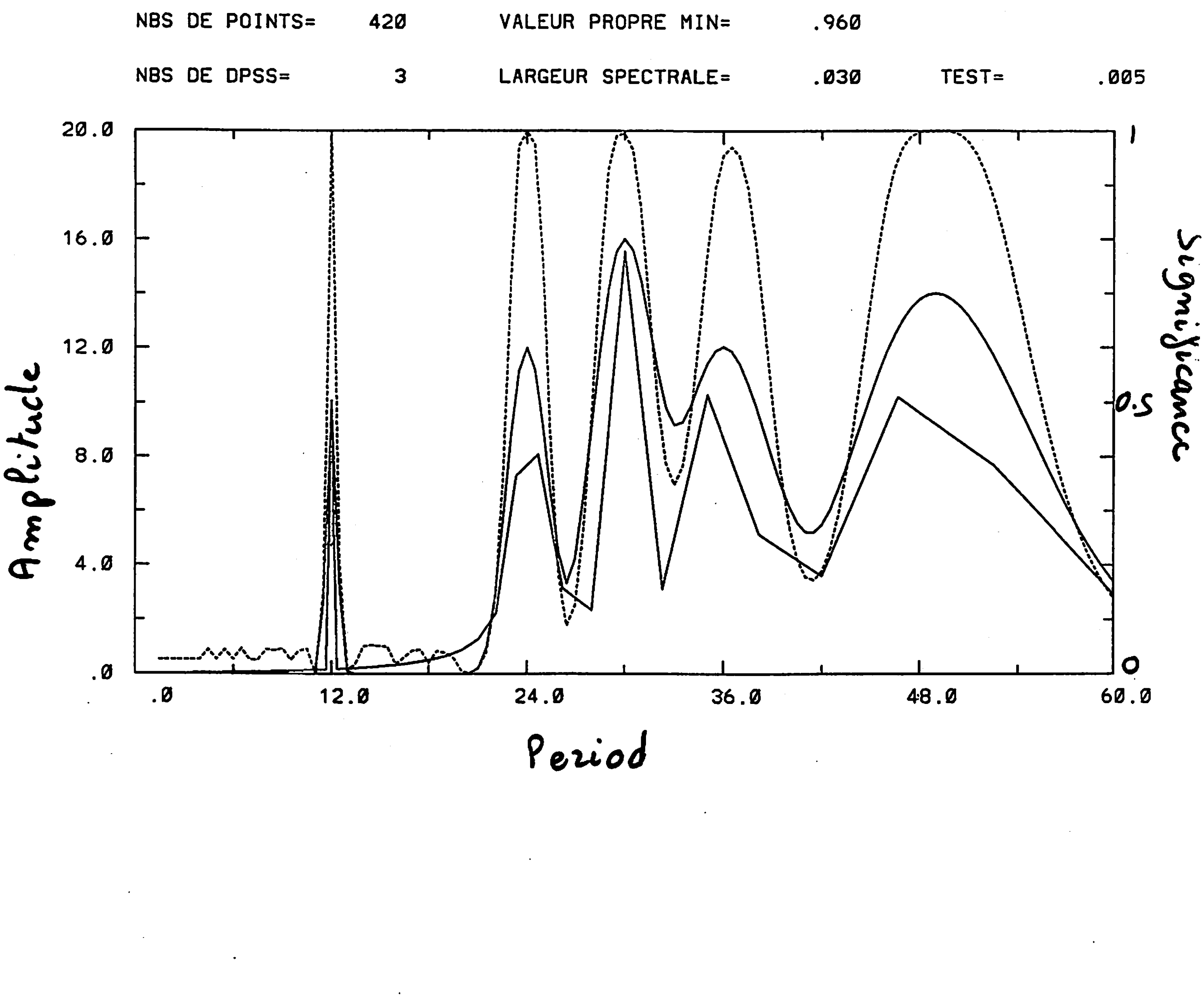
$$y(t) = 10\cos\left(\frac{2\pi}{12}t + \frac{\pi}{3}\right) + 12\sin\left(\frac{2\pi}{24}t\right) + 16\sin\left(\frac{2\pi}{30}t + \frac{\pi}{4}\right)$$

$$+ 12\sin\left(\frac{2\pi}{36}t + \frac{\pi}{2}\right) + 14\sin\left(\frac{2\pi}{49}t + 3\frac{\pi}{5}\right)$$

t= 510 to 930


NBS DE POINTS= 420
VALEUR PROPRE MIN= .977
NBS DE DPSS= 4
LARGEUR SPECTRALE= .040
TEST= .006
20.0
16.0
12.0
8.0
4.0
.0
.0
12.0
24.0
36.0
48.0
60.0
1
0.5
0
Amplitude
Significance
Period

t= 30 to 450


NBS DE POINTS= 420
VALEUR PROPRE MIN= .960
NBS DE DPSS= 3
LARGEUR SPECTRALE= .030
TEST= .005
20.0
16.0
12.0
8.0
4.0
.0
.0
12.0
24.0
36.0
48.0
60.0
1
0.5
0
Amplitude
Significance
Period

wavelets

- Fourier transform:

$$S(f) = \int_{-\infty}^{+\infty} s(t)\, e^{i\omega t}\, dt$$

- time-frequency representation:

$$C_{a,b} = \int_{-\infty}^{+\infty} s(t)\, \psi_{a,b}(t)\, dt$$

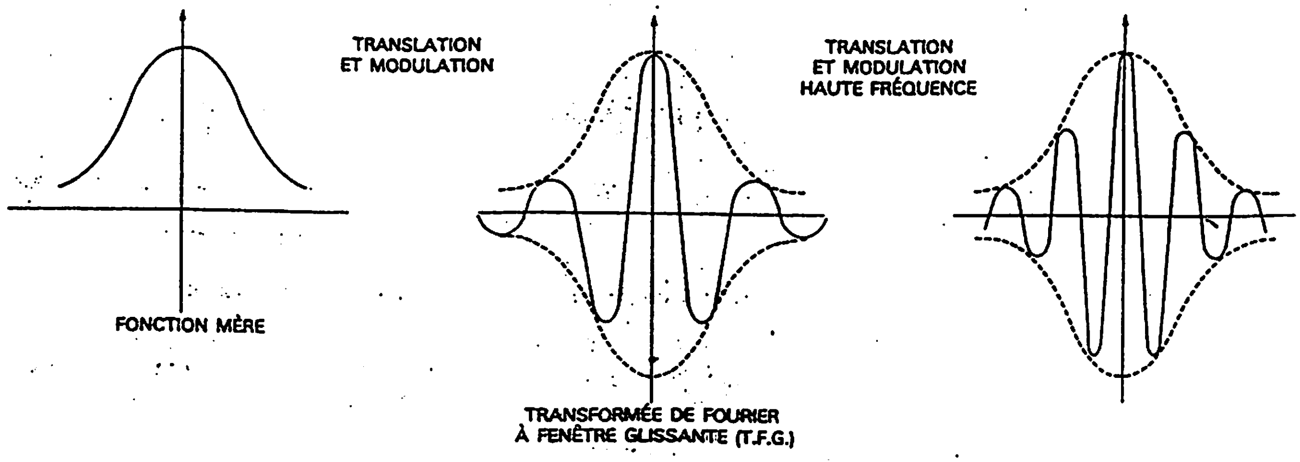
$a \rightarrow$ frequency
$b \rightarrow$ time

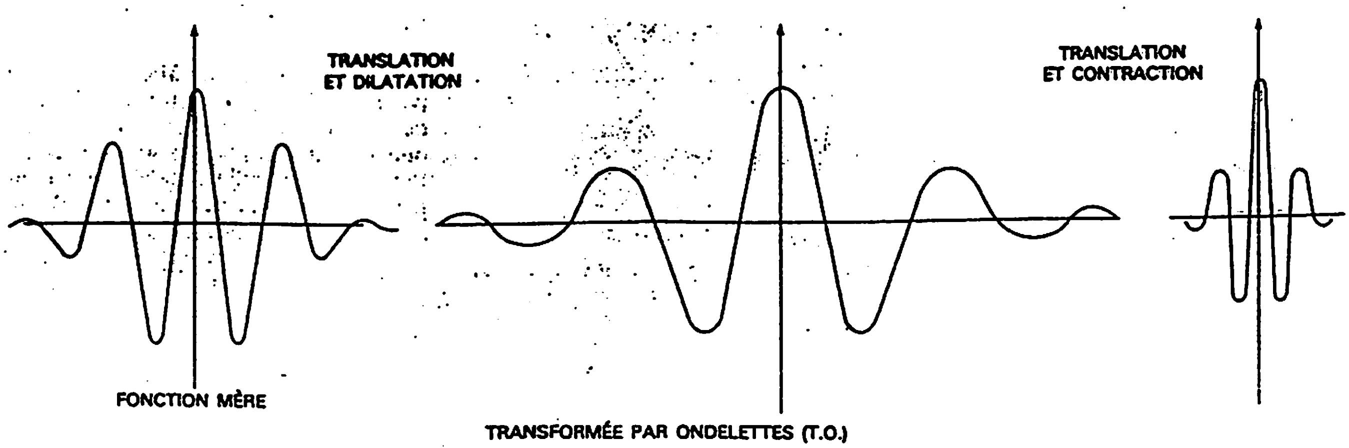
For example sliding window~~ed~~ Fourier transform

$$\psi_{a,b}(t) = e^{iat}\, \phi(t-b)$$

ϕ : time window

first: Gabor (1940), gaussian window


FONCTION MÈRE
TRANSLATION
ET MODULATION
TRANSFORMÉE DE FOURIER
À FENÊTRE GLISSANTE (T.F.G.)
TRANSLATION
ET MODULATION
HAUTE FRÉQUENCE


FONCTION MÈRE
TRANSLATION
ET DILATATION
TRANSFORMÉE PAR ONDELETTES (T.O.)
TRANSLATION
ET CONTRACTION

wavelets = special type of $\psi_{a,b}$

It exists a large variety of wavelets!

- having different properties
- leading to different decomposition

Here, use of multiresolution analysis:

It exist $\psi \longrightarrow \psi_{a,b}$

ψ localized $\longrightarrow$ $\underset{|t| \to \infty}{\psi \to 0}$ quickly

ψ 'wavelike' $\longrightarrow$ ψ oscilating function

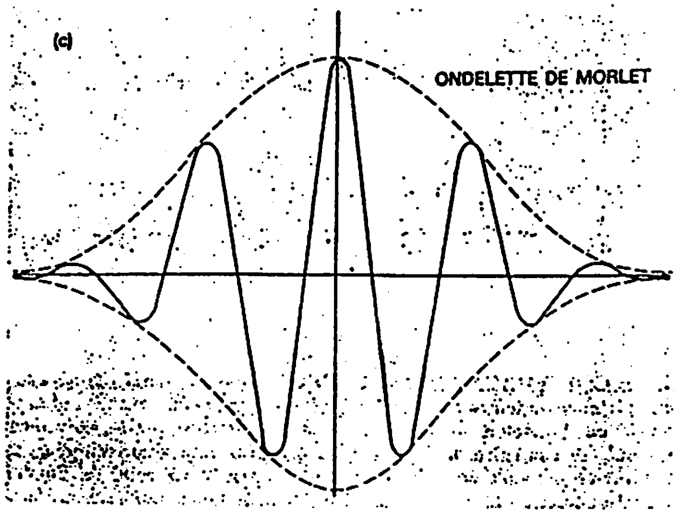
$\overline{\psi} \equiv 0$

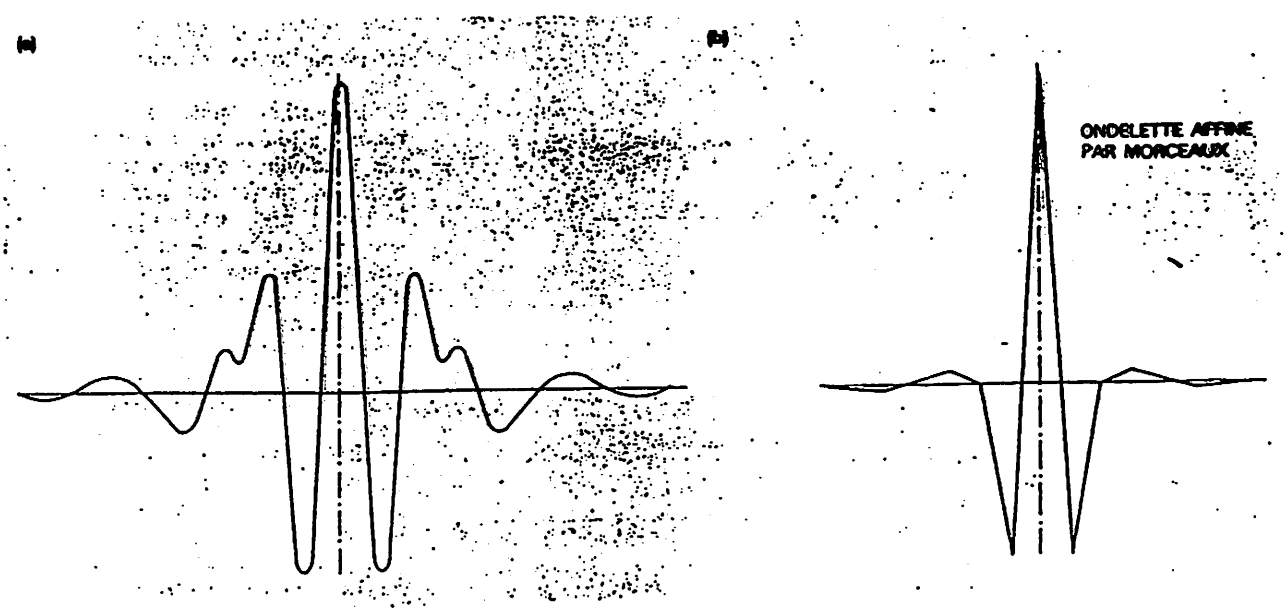
$$\int_{-\infty}^{\infty} \psi(t) = \cdots = \int_{-\infty}^{\infty} t^m \psi(t) = 0$$

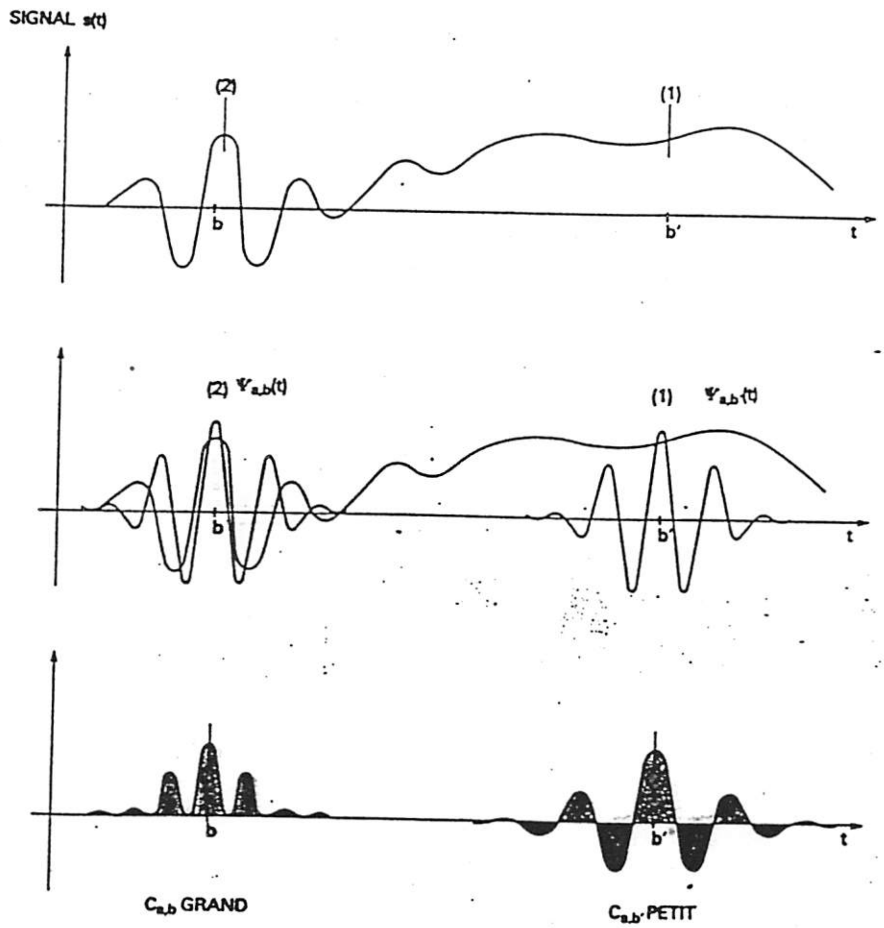
Morlet wavelet:

$$\psi = \cos(5t)\, e^{-t^2/2}$$

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{t-b}{a}\right)$$

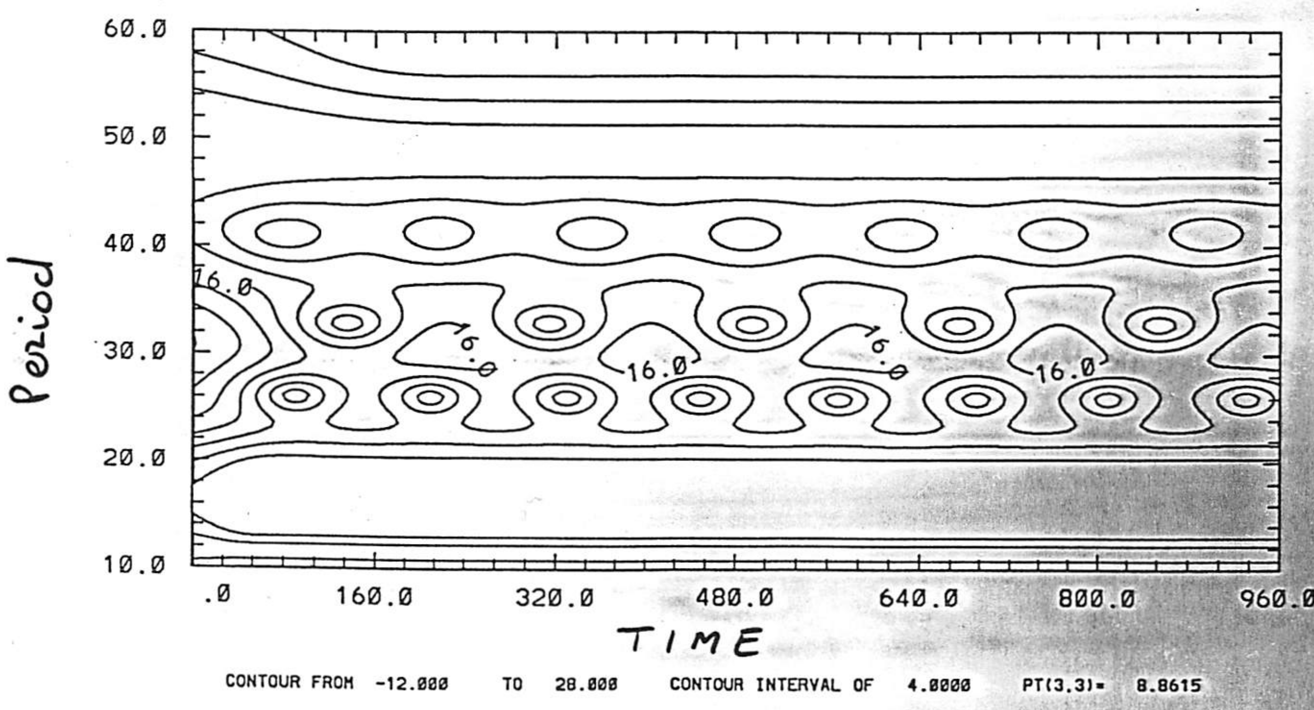

(c)
ONDELETTE DE MORLET


ONDELETTE AFFINE
PAR MORCEAUX

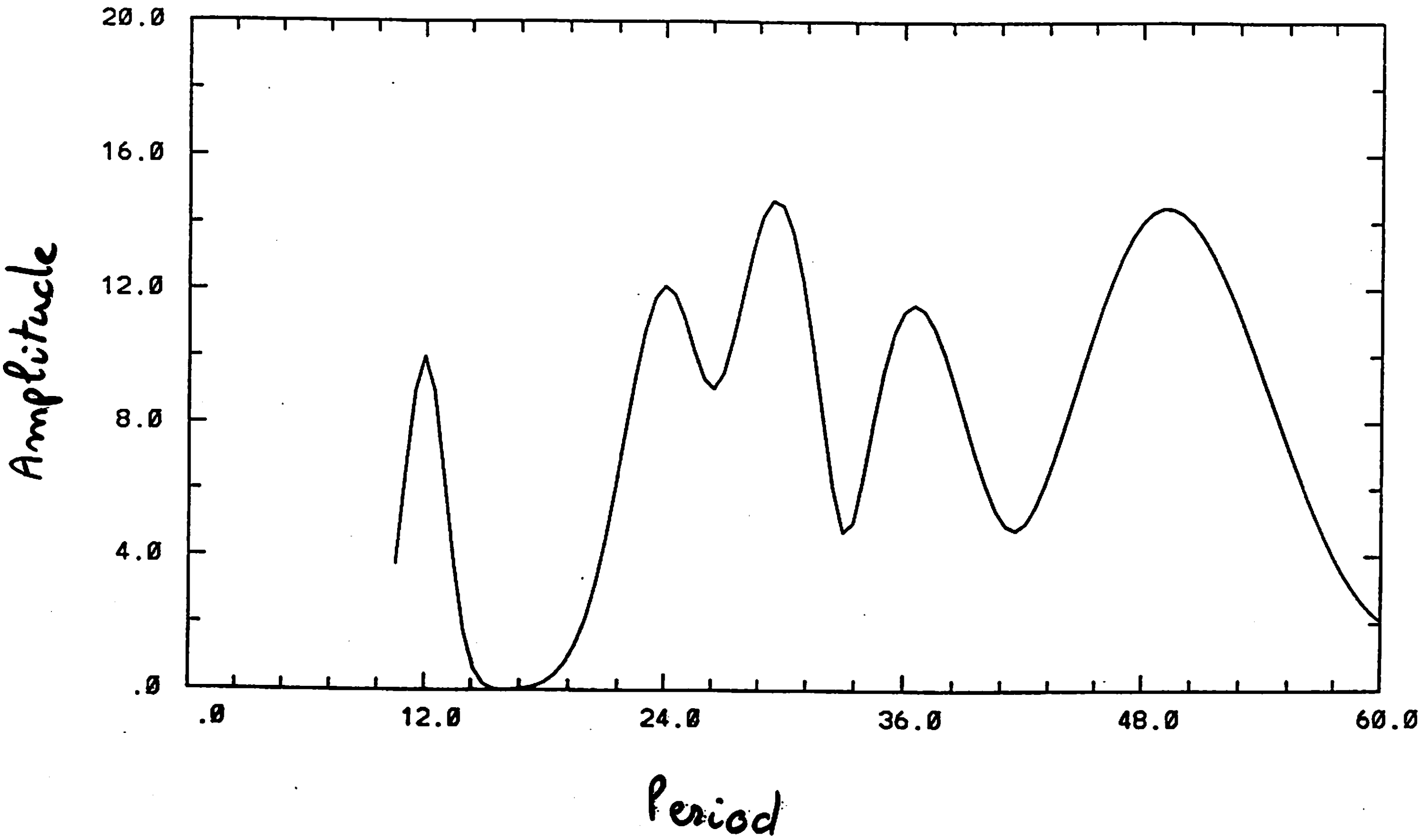


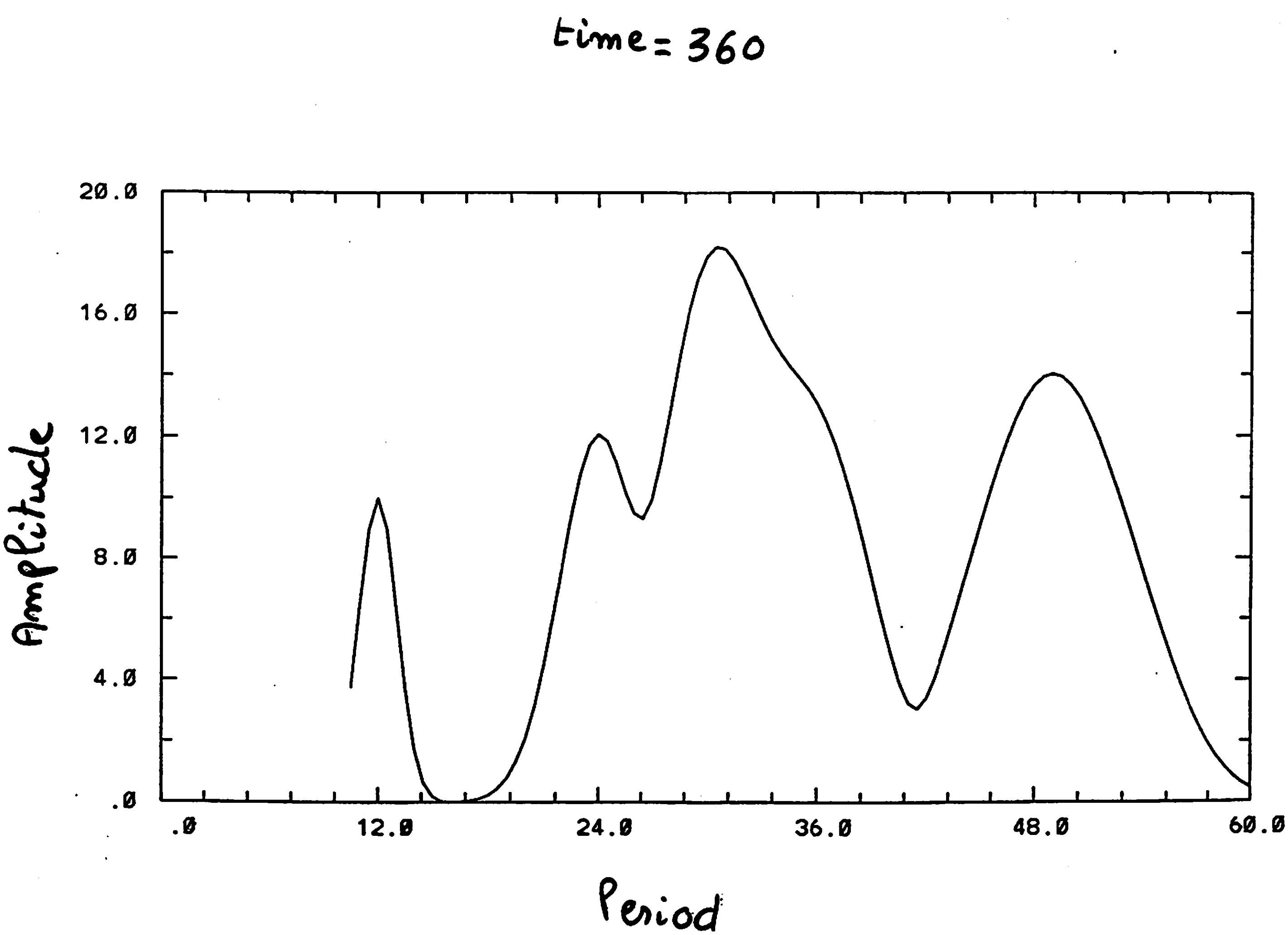
SIGNAL s(t)
(2)
(1)
b
b'
t
(2) $\Psi_{a,b}(t)$
(1) $\Psi_{a,b'}(t)$
b
b'
t
b
b'
t
$C_{a,b}$ GRAND
$C_{a,b'}$ PETIT

Same serie as for Multi-taper

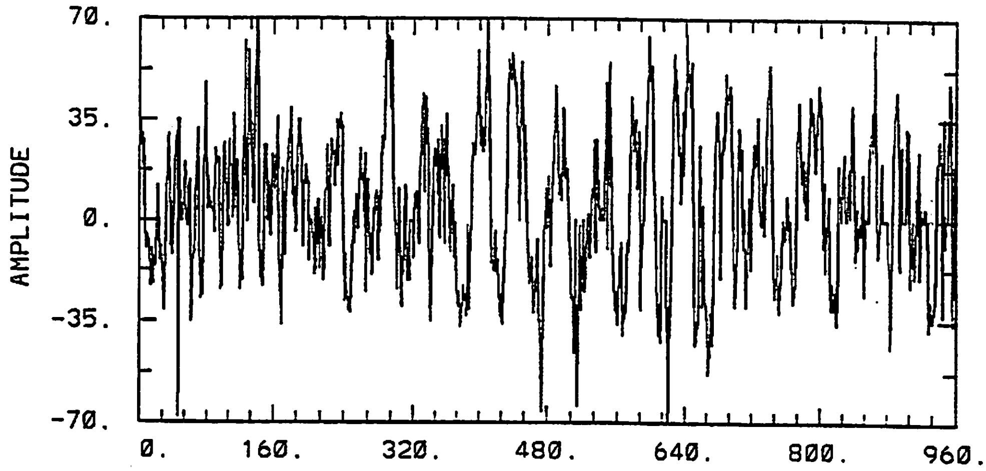


time= 240




time = 360
Amplitude
Period
20.0
16.0
12.0
8.0
4.0
.0
.0
12.0
24.0
36.0
48.0
60.0

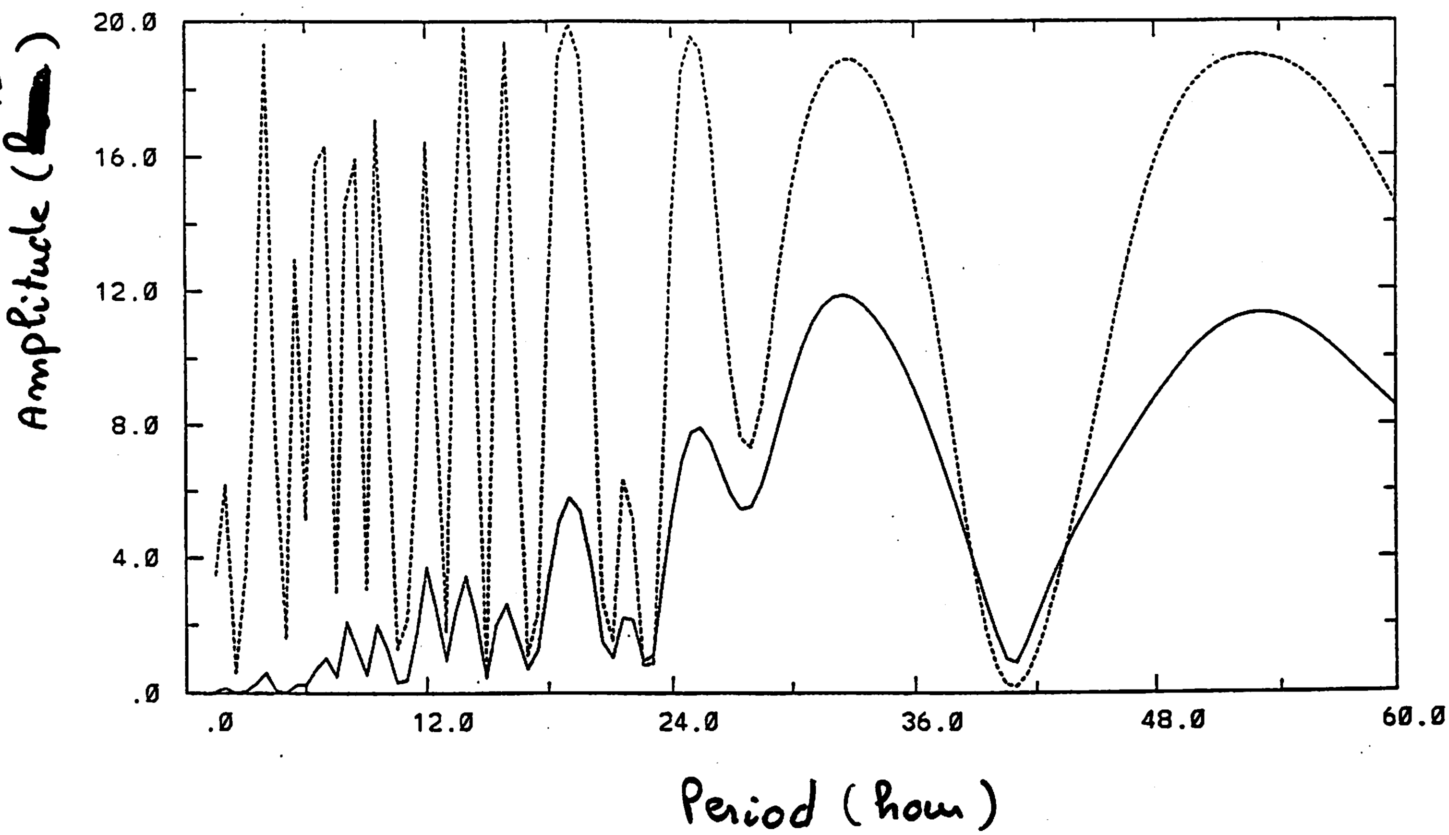
WIND



V 1/2

NBS DE POINTS= 410 VALEUR PROPRE MIN= .970

NBS DE DPSS= 4 LARGEUR SPECTRALE= .040 TEST= .005

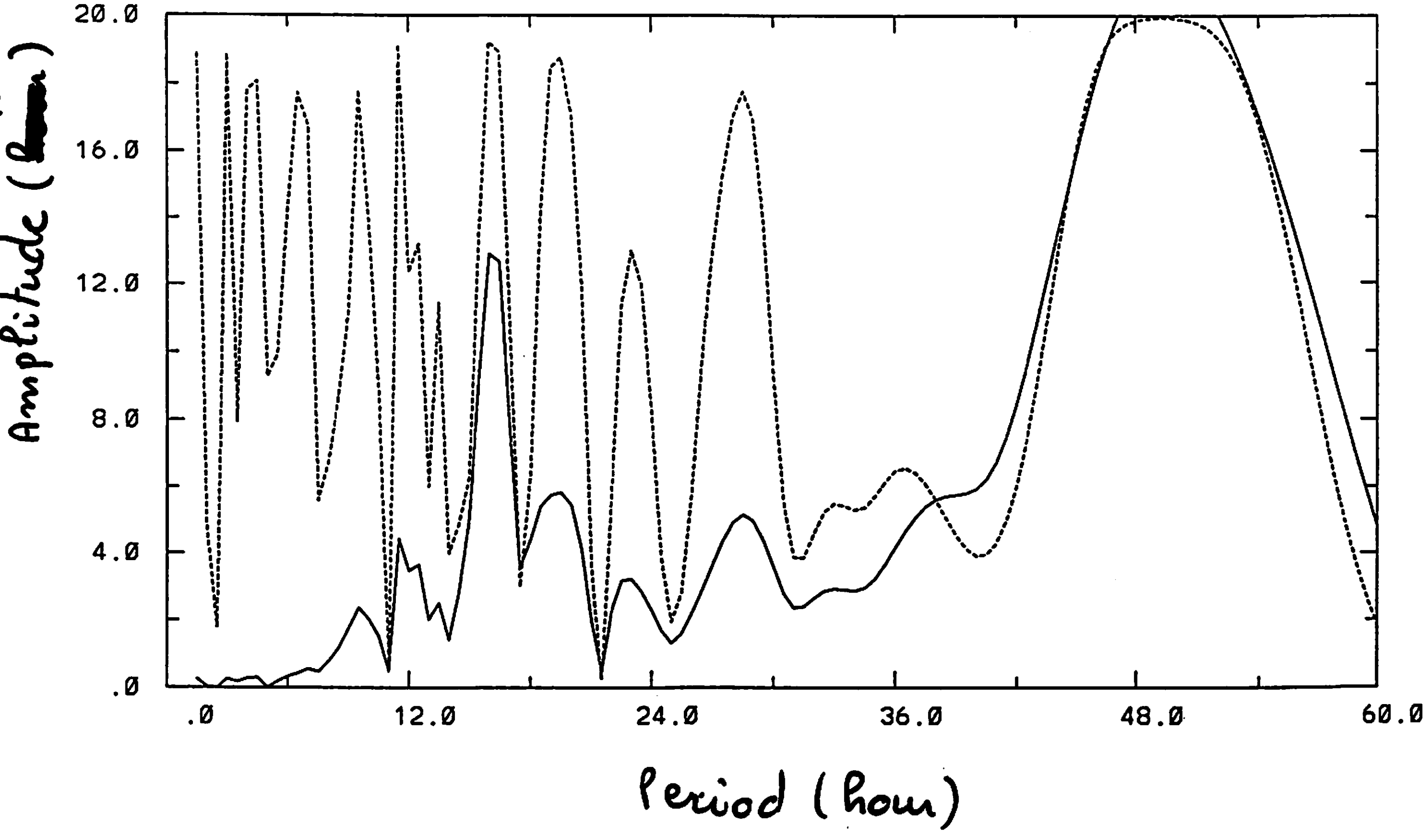


t= 35 to 445 hours

V 2/2

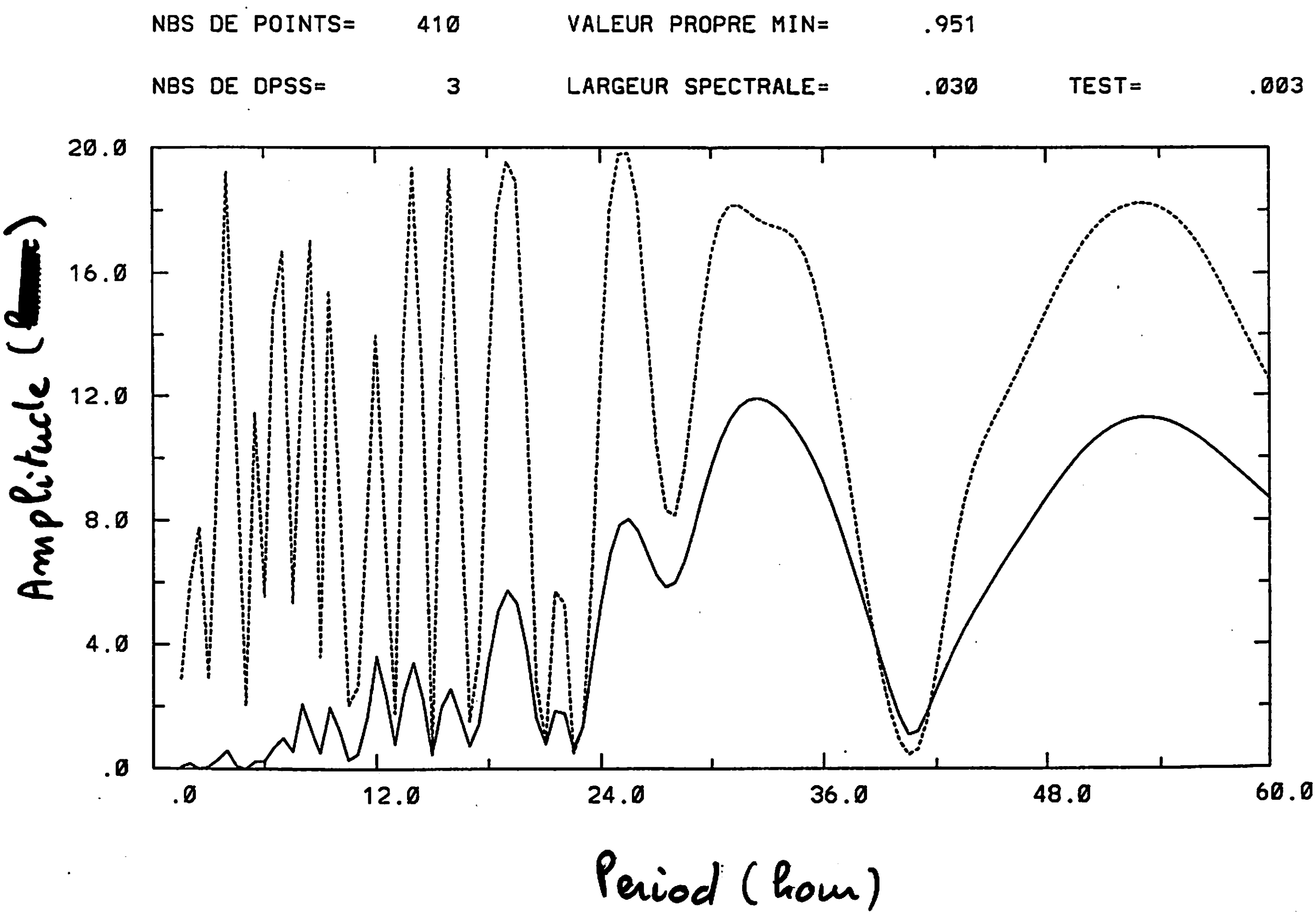
NBS DE POINTS= 410 VALEUR PROPRE MIN= .970

NBS DE DPSS= 4 LARGEUR SPECTRALE= .040 TEST= .005



t= 5Φ5 to 925 hours

V 1/2

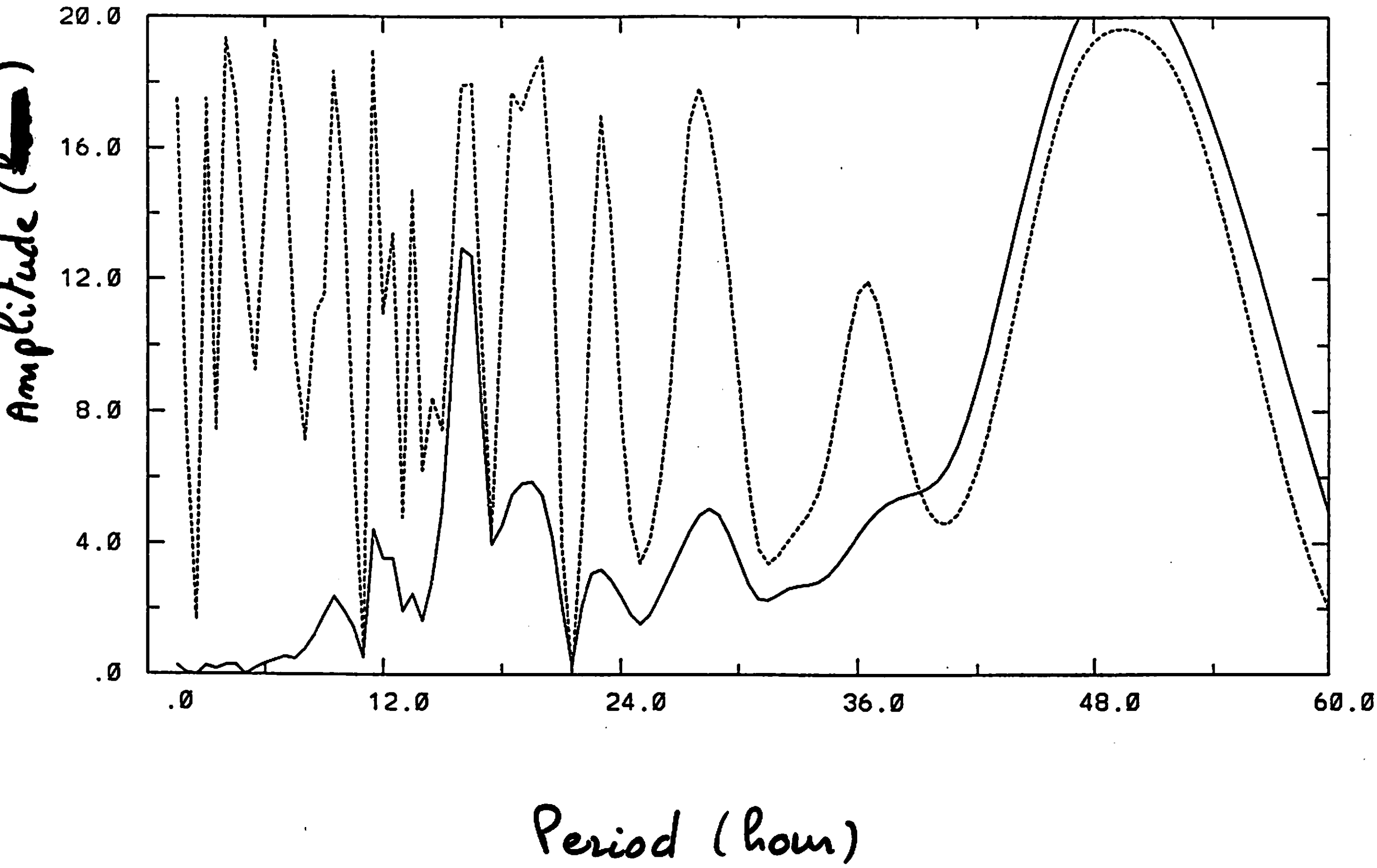


t= 35 to 445 hours

V 2/2

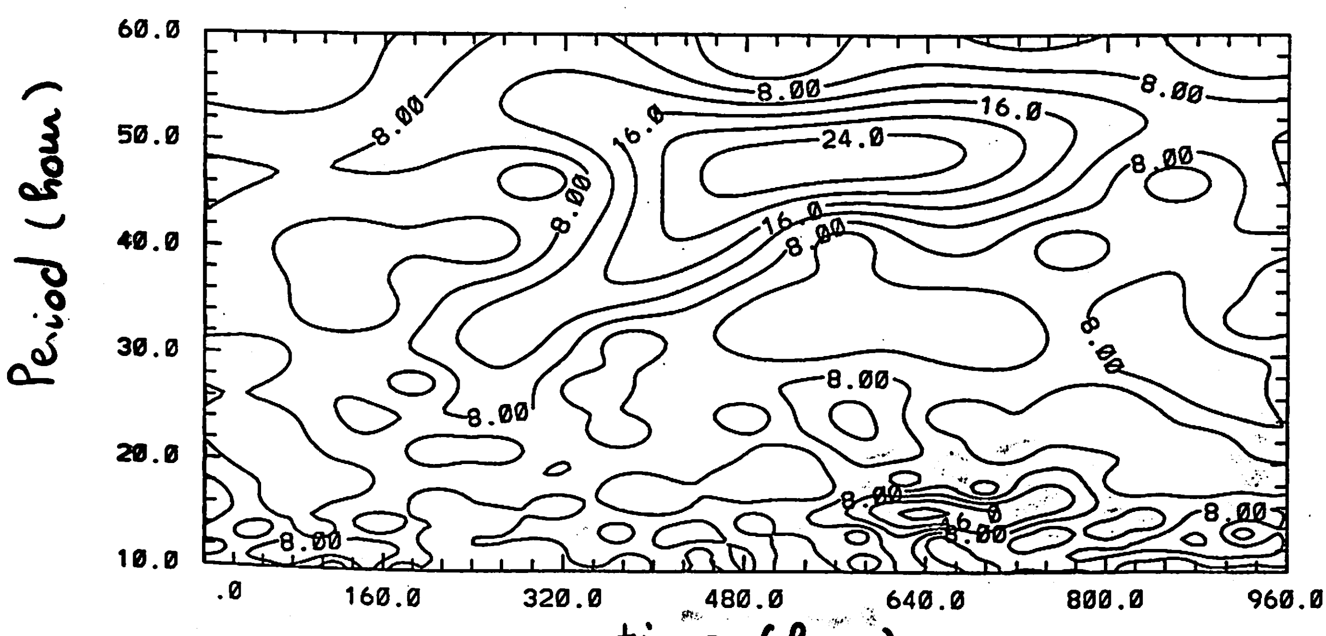
NBS DE POINTS= 410 VALEUR PROPRE MIN= .951

NBS DE DPSS= 3 LARGEUR SPECTRALE= .030 TEST= .003



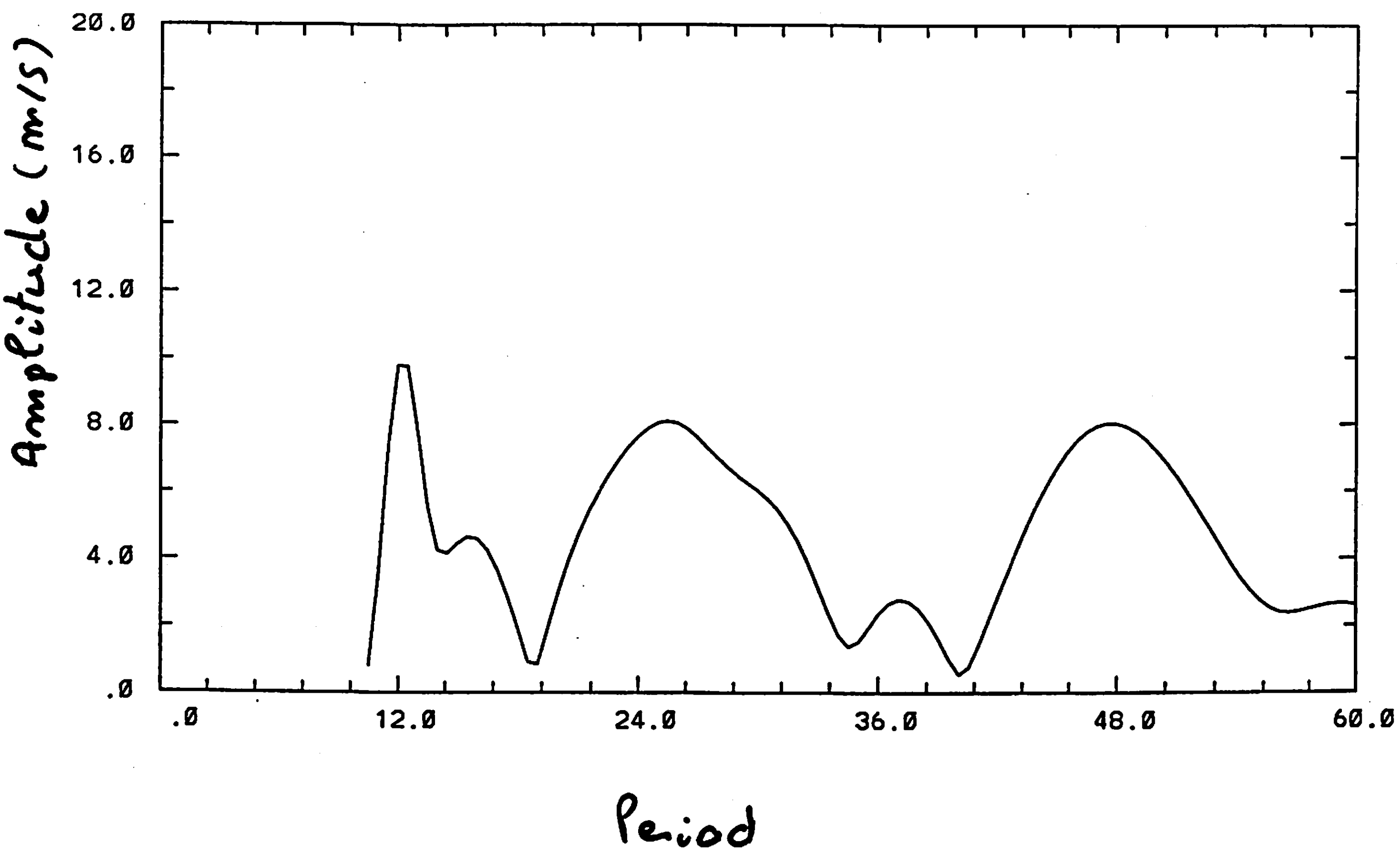
t = 515 to 925 hours

Meridional Wind

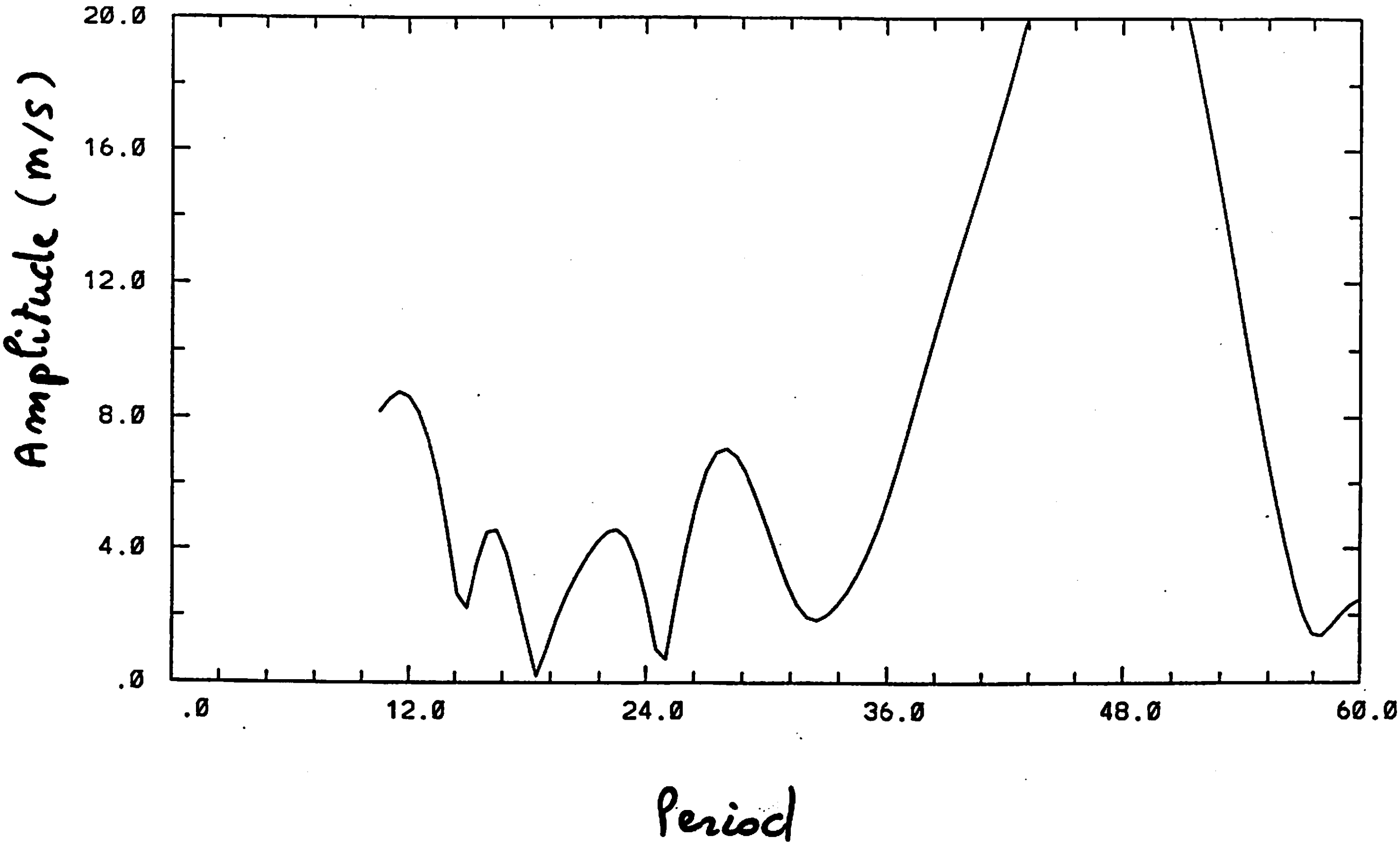


CONTOUR FROM .00000E+00 TO 24.000 CONTOUR INTERVAL OF 4.0000 PT(3,3)= 3.0302

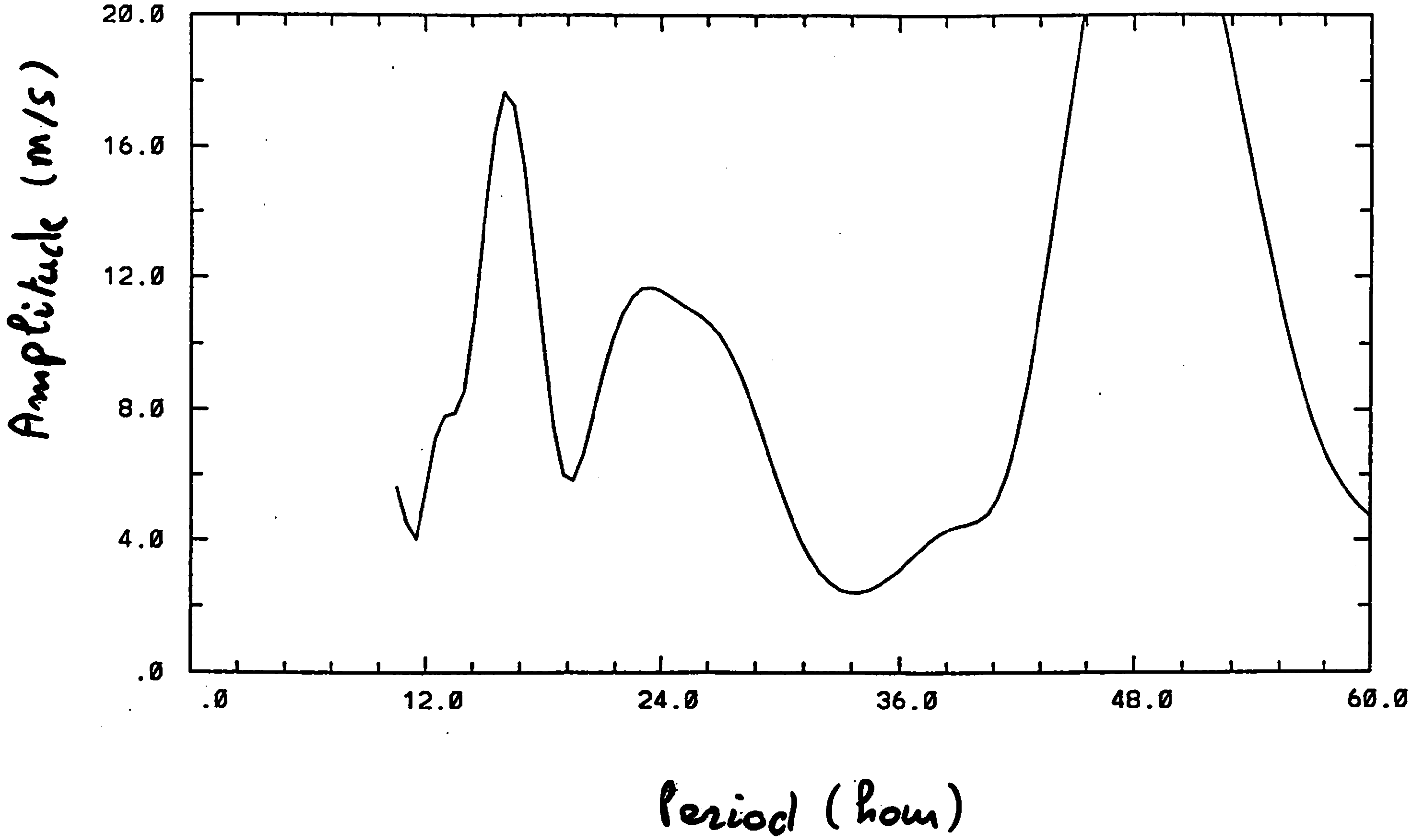
t= 120



t= 480

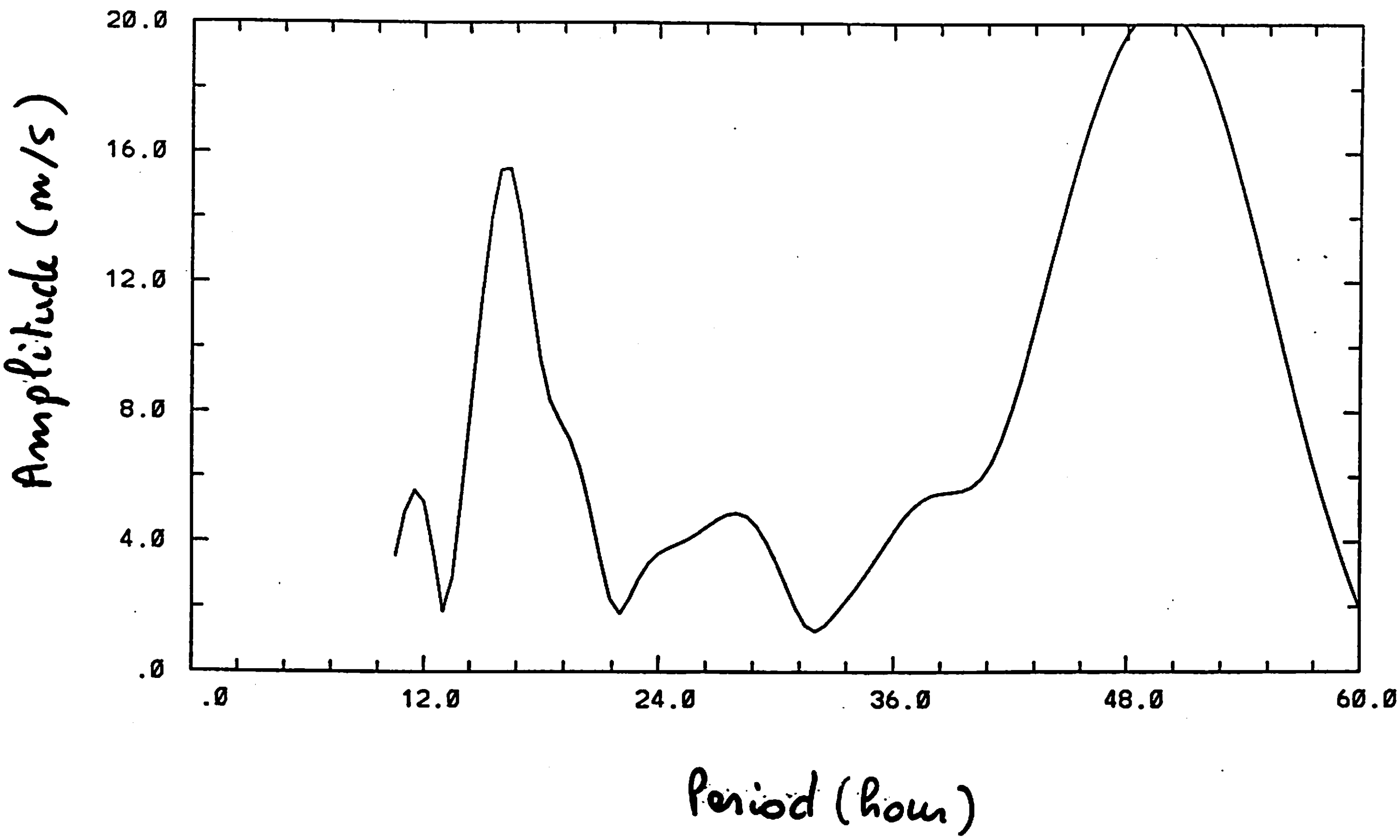


t= 600

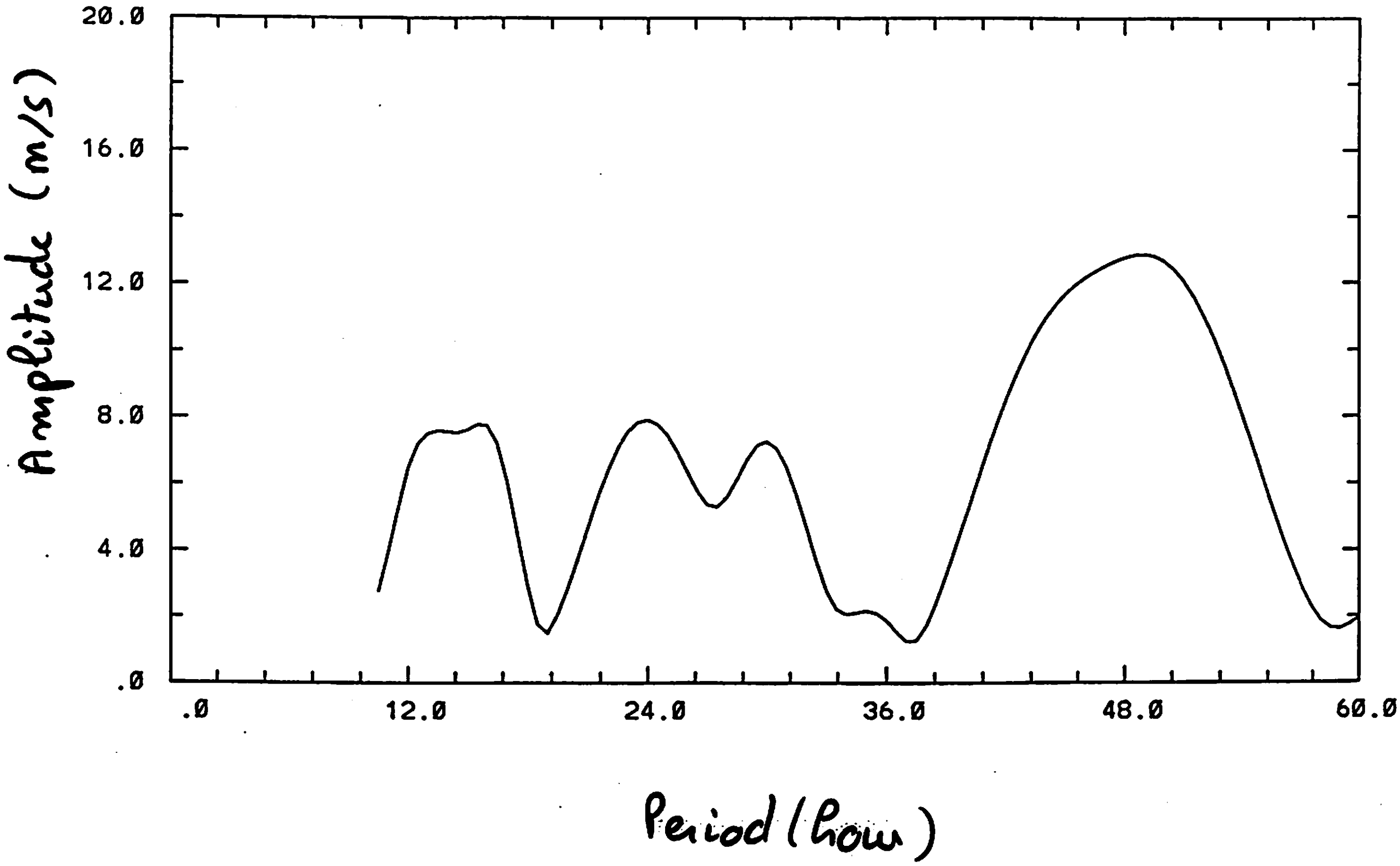


Amplitude (m/s)
20.0
16.0
12.0
8.0
4.0
.0
.0
12.0
24.0
36.0
48.0
60.0
Period (hour)

t = 720


20.0
16.0
12.0
8.0
4.0
.0
Amplitude (m/s)
.0
12.0
24.0
36.0
48.0
60.0
Period (hour)

t= 960


20.0
16.0
12.0
8.0
4.0
.0
Amplitude (m/s)
.0
12.0
24.0
36.0
48.0
60.0
Period (hour)